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Z scaling, fractality, and principles of relativity in the interactions of hadrons and nuclei at high energies

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Abstract

The formation length of particles produced in the relativistic collisions of hadrons and nuclei has relevance to fundamental principles of physics at small interaction distances. The relation is expressed by a z scaling observed in the differential cross sections for the inclusive reactions at high energies. The scaling variable reflects the length of the elementary particle trajectory in terms of a fractal measure. Characterizing the fractal approach, we demonstrate the relativity principles in space with broken isotropy. We derive relativistic transformations accounting for asymmetry of space-time expressed by different anomalous fractal dimensions of the interacting objects.

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1 Introduction

Observation of particles with large transverse momenta produced in high energy collisions of hadrons and nuclei provides unique information about the properties of quark and gluon interactions. As follows from numerous studies in relativistic physics (see e.g. [1, 2, 3]), a common feature of the processes is the local character of the hadron interactions. This leads to a conclusion about dimensionless constituents participating in the collisions. Fact that the interaction is local manifests naturally in a scale-invariance of the interaction cross sections. The invariance is a special case of the 'automodelity' principle which is an expression of self-similarity [4, 5]. This property enables to predict and study various phenomenological regularities reflecting the point-like nature of the underlying interactions. A special role play the deep inelastic processes which confirm the idea of hadrons as composite extended objects with internal degrees of freedom. In the collisions with nuclei the structure was revealed at the sub-nucleon level.

The ideas were implemented into the formulation of the z scaling [6] for description of the inclusive particle production at high energies. The concept of self-similarity of the interactions was complemented by considerations about the fractal character of the objects undergoing the collisions [7]. The goal of the paper is to focus on the general premisses of the z scaling in view of fundamental principles of physics at small interaction distances. It concerns the scale dependence of physical laws gradually emerging in various experimental and theoretical investigations. Such extension of physics is intrinsically linked to the evolution of the concept of space-time. It has been proposed [8, 9] that the topology of space-time becomes complicated with decreasing scales. In the extreme case, when approaching the Plank scale, the structure of space-time is supposed to be extremely irregular (foam-like). The structure is characterized by explicitly scale dependent metric potentials. Asking questions about the metrics leads one to question the relativity. There exists suggestions [10] that the conventional special-relativistic relation between momentum and energy might be modified reflecting 'recoil' background metric changes during the high energy interactions. This in turn could dissolve some cutoff problems in the ultra-relativistic region.

The basic assumption tackled in the paper is the principle of relativity which, besides motion, applies also to the laws of scale. There is one mathematical concept expressing the self-similarity and complying with the scale relativistic approach, so-called fractals [11]. The geometrical objects model the internal parton structure of hadrons and nuclei revealed in their interactions at high energies. The description of the parton interactions as expressed by the construction of the z scaling is presented in Sec. II. Some aspects of the fractality in view of the scaling variable are discussed in Sec. III. In Sec. IV., we present a formalism resulting from application of the relativity principles to the space with broken isotropy which is a characteristic feature of a more fundamental concept of fractal space-time. One of the consequences of the approach is a dispersion law connecting the energy and momentum of a particle having implications to the non-standard relations of the quantities to the particle's velocity. In Sec. V., we apply the results to the kinematic variables determined by the starting assumptions in the definition of the scaling variable z. As a result we obtain the relation between the coefficient expressing the space-time anisotropy induced in the interaction and the ratio of the anomalous (fractal) dimensions of the interacting fractal objects.

2 Z scaling and constituent interactions

We consider the high energy collisions of hadrons and nuclei as an ensemble of individual interactions of their constituents. The constituents can be regarded as partons in the parton model or quarks and gluons which are building blocks in the theory of QCD. The presented approach is substantially based on a premise about fractal character concerning the parton (quark and gluon) content of the composite structures involved. The interactions of the constituents are local relative to the resolution which is a function of the kinematical characteristics of the particles produced in the collisions. The kinematical variables characterize the resolution in terms of the underlying processes. They are determined in a way accounting for maximal relative number of initial parton configurations which can lead to the production of the observed secondaries. In the case of the single-particle inclusive production, the construction was presented in Ref. [7]. It reflects the principles of locality, self-similarity and fractality which govern the collisions of hadrons and nuclei at high energies.

The locality principle is widely accepted and follows from numerous experimental and theoretical investigations. In accordance with this principle, it has been suggested [12] that gross features of the single-inclusive particle distributions for the reaction

$$M_1 + M_2 \rightarrow m_1 + X \tag{1}$$

can be described in terms of the corresponding kinematical characteristics of the interaction

$$(x_1M_1) + (x_2M_2) \rightarrow m_1 + (x_1M_1 + x_2M_2 + m_2). \tag{2}$$

The M_1 and M_2 are masses of the colliding hadrons (or nuclei) and m_1 is the mass of the inclusive particle. The parameter m_2 is used in connection with internal conservation laws (for isospin, baryon number, and strangeness). The x_1 and x_2 are the scale-invariant fractions of the incoming four-momenta P_1 and P_2 of the colliding objects. Considering the process (2) as a local collision of the constituents, we have exploited the coefficient

$$\Omega(x_1, x_2) = m^2(1-x_1)^{\delta_1}(1-x_2)^{\delta_2} \tag{3}$$

as a quantity proportional to the number of all initial constituent configurations which can lead to the production of the inclusive particle m_1. Here m is a mass constant (its typical value being the nucleon mass). The coefficient connects the kinematical characteristics of the elementary interaction with global parameters of the colliding hadrons (or nuclei). It reflects the fractal character of the structural objects revealed by the interaction. The parameters δ_1 and δ_2 relate to their anomalous (fractal) dimensions.

The momentum fractions x_1 and x_2 are determined in the way to maximize the value of Ω, accounting simultaneously for the minimum recoil mass hypothesis [12] in the constituent interaction. The hypothesis states that the interaction is a binary collision subjected to the condition

$$(x_1P_1 + x_2P_2 - q)^2 = (x_1M_1 + x_2M_2 + m_2)^2, \tag{4}$$

where q is the four-momentum of the inclusive particle with the mass m_1. The fractions resulting from these requirements were presented in Ref. [7]. They have the form

$$x_1 = \lambda_1 + \chi_1, \qquad x_2 = \lambda_2 + \chi_2, \tag{5}$$

where

$$\lambda_1 = \frac{(P_2q)+M_2m_2}{(P_1P_2)-M_1M_2}, \qquad \lambda_2 = \frac{(P_1q)+M_1m_2}{(P_1P_2)-M_1M_2}. \tag{6}$$

The χ_i can be expressed as follows

$$\chi_1 = \sqrt{\mu_1^2+\omega_1^2}-\omega_1, \qquad \chi_2 = \sqrt{\mu_2^2+\omega_2^2}+\omega_2, \tag{7}$$

where

$$\mu_1^2 = \alpha\lambda^2\frac{(1-\lambda_1)}{(1-\lambda_2)}, \qquad \mu_2^2 = \frac{\lambda^2}{\alpha}\frac{(1-\lambda_2)}{(1-\lambda_1)}, \tag{8}$$

$$\omega_1 = \frac{(\alpha-1)}{2}\frac{\lambda^2}{(1-\lambda_2)}, \qquad \omega_2 = \frac{(\alpha-1)}{2\alpha}\frac{\lambda^2}{(1-\lambda_1)}, \tag{9}$$

with

$$\lambda = \sqrt{\lambda_1\lambda_2+\lambda_0}, \qquad \lambda_0 = \frac{0.5(m_2^2-m_1^2)}{(P_1P_2)-M_1M_2}. \tag{10}$$

Both x_1 and x_2 consist of two terms. The first terms, λ_i, represent the leading parts of the interacting constituents and do not depend on δ_1 and δ_2. The second terms are functions of the parameter α, where $\alpha = \delta_2/\delta_1$ is the ratio of the corresponding anomalous (fractal) dimensions of the objects undergoing the interaction. It characterizes a relative state of the scale of the reference system and is given by the type of colliding hadrons or nuclei. The terms χ_i represent some sort of a parton 'coat' of the leading parts forming the interacting constituents. They consist of a net of tiny partons forming a fractal structure. The energy, momentum and mass of the parton coat depends on the scale of the reference system characterized by the value of α. According to the decomposition, the binary subprocess (2) can be rewritten to the symbolic form

$$(\lambda_1+\chi_1)+(\lambda_2+\chi_2) \to (\lambda_1+\lambda_2)+(\chi_1+\chi_2). \tag{11}$$

The variables χ_i correspond to the parts of the interacting constituents which are responsible for the creation of the recoil $(x_1M_1+x_2M_2+m_2)$. In high energy collisions, one can, in principle, recognize the interaction of the constituents which underlies the inclusive particle production at high energies. The level of the recognition is given by a resolution ε^{-1}. The resolution should depend on the extensive parameters of the interacting system, on the available energy and should be a function of the kinematical characteristics of the observed secondaries. The extensive parameters in the nucleus-nucleus collisions are the atomic numbers A_1 and A_2. Results of our analysis on pA interactions show [7] that the anomalous (fractal) dimensions of nuclei are given by the relation $\delta_A = A\delta$. According to the additive property, the coefficient $\alpha \equiv \delta_2/\delta_1 = A_2/A_1$ is the function of the extensive characteristics of the system but does not depend on the structure parameter δ. Consequently, the kinematical variables x_i do not depend on δ. This property enables us to define the resolution ε^{-1} by means of

$$\varepsilon(x_1,x_2) \equiv (1-x_1)^{A_1}(1-x_2)^{A_2} \sim \Omega^{1/\delta}, \tag{12}$$

where x_1 and x_2 are subjected to the constrain (4). The $\varepsilon(x_1,x_2)$ is the relative uncertainty with which one can single out the binary subprocess (2) from the system of two colliding nuclei. The uncertainty is related to the relative number of all initial configurations of the constituents (given by Ω), interactions of which can lead to the production of the inclusive

particle m_1. The relative uncertainty in the determination of the underlying subprocess can be reduced to smaller subsystems, if the momentum fractions are not too large. Specially, for the interactions involving only single nucleons, one usually introduces the momentum fractions of the interacting nuclei expressed in units of the nucleon mass, $\bar{x}_i = A_i x_i$. In the single nucleon interaction regime, the relative uncertainty can be approximated as follows

$$\varepsilon(x_1, x_2) \equiv (1 - \bar{x}_1/A_1)^{A_1}(1 - \bar{x}_2/A_2)^{A_2} \sim (1 - \bar{x}_1)(1 - \bar{x}_2). \tag{13}$$

On the other hand, the factorization does not apply for the processes in which $\bar{x}_i > 1$. They are known as cumulative processes [4, 12, 13] and correspond to the joining of partons from different nucleons of nuclei. This region is interesting from the point of view of fractality at small scales. Here we focus on small interaction distances under the condition that simultaneously large amount of energy is deposited in it. In the single-particle inclusive reactions, the cumulative process corresponds to larger spatial resolutions determined by the dimensions of the interaction region of the constituent collision. The resolution increases with the momentum transfer in the underlying subprocesses and is influenced by the types of the colliding objects. We discuss this connection in Appendix A in more detail.

The second basic ingredient of the z scaling scheme is the self-similarity principle. In high energy physics, the principle is reflected by the property that the underlying production processes are similar. It results in dropping of certain quantities or parameters out of the physical picture of the interactions. The particle production from self-similar processes can be described in terms of independent variables. Construction of the variables depends on the type of the inclusive reaction. For single-particle inclusive production at high energies, the scaling function [7]

$$H(z) \equiv \frac{\psi(z)}{2\pi z} = \frac{s_{tot}}{\rho_{tot}\sigma_{inel}} \left(\frac{z}{\lambda_2} \frac{\partial z}{\partial \lambda_1} + \frac{z}{\lambda_1} \frac{\partial z}{\partial \lambda_2} \right)^{-1} E\frac{d^3\sigma}{dq^3} \tag{14}$$

was studied in dependence on the single variable z. Here s_{tot} is a square of the total center-of-mass energy and σ_{inel} is the inelastic cross section of the objects M_1 and M_2. The relation connects the inclusive differential cross section and the multiplicity density $\rho_{tot}(s_{tot}, \eta) \equiv d\langle N\rangle/d\eta$ of the corresponding pseudorapidity distribution with the scaling function. The function $\psi(z)$ is interpreted as the probability density to form particles characterized by the value of z. The question is to define such z that could reflect general pattern of the particle production mechanism. The variable was chosen accounting for the locality, self-similarity and fractality as common principles of the particle production at high energies [7]. It represents the fractal measure attributed to the inclusive particle and is proportional to its formation length. The measure consists of a finite part z_0 and a divergent factor depending on the resolution ε^{-1}. It is defined in the form

$$z = z_0\ \varepsilon^{-\delta}, \tag{15}$$

where

$$z_0 = \frac{\hat{s}_{\perp kin}^{1/2}}{m^2 \rho(s)}, \qquad \varepsilon(x_1, x_2) \equiv (1 - x_1)^{A_1}(1 - x_2)^{A_2}. \tag{16}$$

The finite part z_0 is characterized by the transverse kinetic energy

$$\hat{s}_{\perp kin}^{1/2} = \hat{s}_\lambda^{1/2} + \hat{s}_\chi^{1/2} - m_1 - (M_1 x_1 + M_2 x_2 + m_2) \tag{17}$$

of the subprocess (2). It includes the terms

$$\hat{s}_\lambda^{1/2} = \sqrt{(\lambda_1 P_1 + \lambda_2 P_2)^2}, \qquad \hat{s}_\chi^{1/2} = \sqrt{(\chi_1 P_1 + \chi_2 P_2)^2}, \tag{18}$$

representing the transverse energy of the inclusive particle and its recoil, respectively. The factor $\rho(s) \equiv d\langle n \rangle / d\eta|_{\eta=0}$ is the average multiplicity density of the charged particles produced in the central region of the corresponding nucleon-nucleon interaction. The multiplicity density $\rho(s)$ depends on the total center-of-mass energy and includes the dynamical ingredient of the scaling. The divergent part of the variable z is given as a δ power of the resolution ε^{-1}. The parameter δ represents anomalous (fractal) dimension of the trajectories followed by the inclusive particles produced in the high energy collisions of hadrons and nuclei. We consider the anomalous dimension δ to be a characteristic reflecting the intimate structure of particle motion at small scales and thus related to the fractal properties of space-time.

The comparison with experimental data shows that the z scaling represents regularity valid in nucleon-nucleon and proton-nucleus collisions in a wide range of the center-of-mass energy, the detection angle θ and the atomic number A. All parameters in the z scaling scheme are given in terms of the measurable quantities except one, which is the parameter δ. Restriction on this parameter is given by experimental data. We stress that energy and angular independence of the z scaling can be achieved simultaneously by the same value of δ. We have obtained $\delta \sim 0.8$ [7] for the inclusive production of charged particles (pions). The A-universality of the z scaling was demonstrated with the same value as well. The simultaneous energy and angular independence of z scaling for the inclusive productions of jets implies [14] $\delta \sim 1$. It should be emphasized that the production of jets underlie the hard processes with large values of z. In this region the scaling function manifest a clear power law $\psi(z) \sim z^{-\beta}$. Similar behaviour is seen for the inclusive production of single hadrons [7] in the tail of the spectrum. The power law of the scaling function stresses the fractal attributes of the processes which are preferable to study mainly in the region of large z. In contrast to this, the scaling function has different shape for small values of z. It corresponds to medium and low transverse momenta $q_\perp$. The underlying parton interactions have character of the soft processes here. The soft interactions are dominant mainly in both fragmentation regions. In this sense, the soft and hard regime of the particle production is distinguished by different behaviour of the scaling function. The quantitative values of the scaling function should be reproduced using a theory of hadronic interactions, which we thrust is QCD. However difficulties in applying the QCD methods in the non-perturbative regime and the lack of fundamental understanding of hadronization even in an perturbative approximation, limits us to the phenomenology. In our description of the inclusive reactions, we aim at grasping main principles that influence the particle production at small scales. We understand the existence of the z scaling itself as confirmation of the hadron interaction locality, self-similarity, and fractality which possess an universal character.

3 Fractality at small distances

Besides the self-similarity principle, the construction of the z scaling regards also the fractality of the hadron interactions. The fractal properties are manifested especially at small interaction distances. In this region, physics becomes scale dependent and possesses its typical property which is the divergence of elementary quantities such as the self-energy, charges

and so on. In the high energy collisions, one of the relevant physical quantities is the formation length of the produced particles divergent at small scales. According to the z scaling hypothesis, the formation length is proportional to the scaling variable z and the production cross sections depend on it in an universal, energy independent way [14, 15, 16].

Universality plays the crucial role. There exist suggestions [17] that universal properties of matter are attributed to the structure of space-time itself. In the theory of relativity both special and general, this concerns the Lorenz transformation and the curvature of the trajectories reflecting fundamental principles of physics. Free particles are moving along smooth geodesical lines, characteristic for the classical (curved) space-time. The situation becomes different at scales typical for the quantum world of the elementary particles. The common property is the unpredictability of motion at small distances. In this region the particles follow irregular trajectories which become non-differentiable. The geometry of the lines of motion can be attributed to the properties of fractals which are extremely irregular objects fragmented at all scales. As an example one can mention the quantum-mechanical path of a particle in the sense of Feynman trajectories [18].

One of the main characteristics of fractals is the divergence of their measures in terms of the increasing resolution. We illustrate this property by the von Koch curve [19] which is characterized by the fractal measure

$$z_\varepsilon = z_0 \cdot \varepsilon^{D_T - D} \tag{19}$$

representing its length. The length of the curve is a function of the resolution ε^{-1} (see Fig.1.). Relation (19) is typical for various fractals and states how the fractal measure depends on the resolution. Fractal objects are characterized by the topological dimensions D_T which are integer quantities and by the fractal dimensions D acquiring generally non-integer values. The anomalous dimension of a fractal

$$\delta \equiv D - D_T > 0 \tag{20}$$

is positive. It is equivalent to power law divergence of the measure z_ε with the increasing resolution. Von Koch curve illuminates the process of 'fractalization'. The curve has the length

$$z_n = z_0(p/q)^n, \qquad p = 4, \quad q = 3 \tag{21}$$

in the n-th approximation. It is composed of p^n segments, each of the length $z_0 q^{-n}$. The measure can be rewritten to the form

$$z_n = z_0 (q^{-n})^{1 - \ln p / \ln q}, \tag{22}$$

which gives the relation (19) with $D_T = 1$, $D = \ln p / \ln q$, and $\varepsilon = q^{-n}$. The anomalous dimension of the von Koch curve δ is positive and thus, with increasing resolution, its length tends to infinity.

These concepts find application in the world of physics at small distances. The fractal character in the initial state reflects the parton (quark and gluon) composition of hadrons and nuclei and reveals itself with a larger resolution at high energies. According to this picture, the collisions of the constituents take place on the fractal background of the interacting objects. The essential assumption concerning the interpretation of the ideas represented by the z scaling hypothesis is expressed in the following statement: *Presence of the interacting fractal objects deforms the structure of surrounding space at small distances. As a consequence, space-time*

becomes locally non-differentiable, fractal with geodesical lines acquiring an extremely irregular scale-dependent shape. The notion of fractal space-time was used in Refs. [17, 20] and its properties have been studied by others [22]. Distortion of the surrounding space-time in the interactions with recoil induce a non-trivial off-diagonal terms in the metric changing the relations between energy and momentum [10]. The issues concern modelling of various aspects of quantum fluctuations which influence particle production and their interactions at small scales.

The secondary partons, produced in fractal space-time follow erratic and scale-dependent geodesics starting from the regions they have been created. During the initial phase of the motion, one can imagine the trajectories as fractal curves similar to the von Koch curve depicted in Fig.1. Formation of a particle from the bare parton realizes along the trajectory characterized by its length. The produced parton, ancestor of the secondary particle, interacts with the vacuum or the surrounding matter field acquiring simultaneously some type of the parton 'coat'. The number of the enveloping partons forming the coat results in an effective increase of the ancestor's dimensions simultaneously changing its mass. During the particle formation process, the path of the leading parton becomes gradually smoother in comparison to that following immediately the instant of the constituent interaction. In the final stage of the process, the relative length of the particle's path is negligible with respect to its value in the very beginning of the motion. It is a consequence of the transition from small scales, characterized by an extremely irregular fractal-like character of the trajectory, to scales larger than, say, the corresponding de Broglie length. Thus, in the collisions of the structural objects such as hadrons or nuclei, the length of the trajectory of a produced particle can increase infinitely with the decreasing scale at which the particle was created.

Let us look at the problem in view of the models describing the interaction of the constituents as a parton-parton collision with subsequent formation of a string stretched between the two parts in the final state of the process (2). Following the same geodesical trajectory, the string is a fractal object with the scale dependent properties. In this picture, the value of Ω (3) can be considered as a quantity reflecting the tension of the string. The string tension coefficient is characterized by the density of the straight-like segments along its length. For all resolutions, the product of the coefficient and the length of the string z_ε represents the finite quantity

$$\hat{s}_h^{1/2} = \Omega \cdot z_\varepsilon, \tag{23}$$

which is the energy of the string. The form of the tension coefficient, as presented by Eq. (3), accounts for the shape of the string deformed in consequence of the fractal structure of space-time. With the increasing resolution, the string is more and more fragmented and the deformations result in the diminishing of its tension. Physically, the deformations of the string can be caused by fluctuations of the QCD vacuum disturbed in presence of the interacting fractal objects at small distances. The string tension

$$\Omega \sim \varepsilon^\delta \tag{24}$$

is a function of the resolution ε^{-1} and is characterized by the anomalous (fractal) dimension δ of the string. The resolution corresponds to a characteristic size of the space-time region of the interaction and depends on the momentum fractions carried by the interacting constituents. In the determination of the fractions, we consider an optimization method dealing with the fractal trajectories of particles at any scale. The optimal trajectory is defined by the condition that,

for the underlying collision of the constituents, the tension of the fractal-like string, stretched out by the produced parton, should by maximal. This is equivalent to the extremum of the expression (3) under the condition (4). The maximal tension of the string is thus given by the minimal value of the resolution and corresponds to the geodesics which are in a sense 'optimal' curves.

The energy of the string connecting the two objects in the final state of the process (2) is given by the energy of the colliding constituents. The string evolves further and splits into pieces. The resultant number of the string pieces is proportional to the number or density of the final hadrons measured in experiment. As known from various experimental and theoretical studies concerning the multiple production, the produced multiplicity is proportional to the excitation of transverse degrees of freedom. Therefore, the string transverse energy is a measure of multiplicity. Such ideas allow us to interpret the ratio

$$\hat{s}_h^{1/2} \equiv \hat{s}_{\perp kin}^{1/2}/\rho(s) \tag{25}$$

as a quantity proportional to the energy of the string piece, which does not split already, but during the formation process converts into the observed secondary. The string splitting is self-similar in the sense that the leading piece of the string forgets the string history and its formation does not depend on the number and behaviour of other pieces. We write the energy of the single piece of the string in the form (23). This is equivalent to the definition of the variable $z \equiv z_\varepsilon$ as a fractal measure proportional to the length of the string piece, or the formation length, on which the inclusive particle is formed. The corresponding scaling function $H(z)$ reflects the evolution of the formation process of the inclusive particle along its fractal-like trajectory of the length z. We add here that there exists also complementary interpretation of the factor Ω. According to the ideas presented in Ref. [7], Ω reflects the relative number of all initial configurations containing the constituents which carry the momentum fractions x_1 and x_2. The number of the configurations in one colliding object is given by the power law characteristic for fractals. In fractal dynamics the resolution ε^{-1} is given by the maximal number of the initial configurations which can lead to the production of the particle m_1.

Generally, the fractal approach to the high energy collisions of hadrons and nuclei needs more profound understanding. It concerns the deformation of space-time at small scales and attributes additional meaning to the physical quantities such as the momentum, mass, energy or velocity. They may be defined from parameters of the fractal objects in terms of the fractal geometry [17]. This includes extension of the relativity principles to the relativity of scales as well as to the more comprehensive scale-motion relativistic concepts.

4 Break down of the reflection invariance, the way towards scale-motion relativity

General solution to the theory of the special relativity is the Lorenz transformation. As demonstrated by Nottale, it can be obtained under minimal number of three successive constraints. They are (i) homogeneity of space-time translated as the linearity of the transformation, (ii) the group structure defined by the internal composition law and (iii) isotropy of space-time expressed as the reflection invariance. Let us consider the relativistic boost along the x-axis. The transformation concerns the variables x and t which refer not only to the coordinate and time, but also to any quantities having the mathematical properties considered.

Without any loss of generality, the linearity of the transformation can be expressed in the form [21]

$$x' = \gamma(u)[x - ut], \tag{26}$$

$$t' = \gamma(u)[A(u)t - B(u)x], \tag{27}$$

where γ, A, and B are functions of a parameter u. The parameter represents usual velocity (in units of the velocity of light c) in the motion relativity or the 'scale velocity' used, e.g., in the concept of the scale relativity concerning fractal dimensions and fractal measures [17]. Let us compose the transformation with the successive one

$$x'' = \gamma(v')[x' - v't'], \tag{28}$$

$$t'' = \gamma(v')[A(v')t' - B(v')x']. \tag{29}$$

The result can be written in the form

$$x'' = \gamma(u)\gamma(v')[1 + B(u)v'] \left[x - \frac{u + A(u)v'}{1 + B(u)v'}t\right], \tag{30}$$

$$t'' = \gamma(u)\gamma(v')[A(u)A(v') + B(v')u] \left[t - \frac{A(v')B(u) + B(v')}{A(u)A(v') + B(v')u}x\right]. \tag{31}$$

The principle of relativity is expressed by the constraint (ii) which is the group structure of the transformations. The condition tells us that Eqs. (30) and (31) keep the same form as the initial ones in terms of the composed velocity

$$v = \frac{u + A(u)v'}{1 + B(u)v'}. \tag{32}$$

The requirement can be satisfied under the following conditions

$$\gamma(v) = \gamma(u)\gamma(v')[1 + B(u)v'], \tag{33}$$

$$\gamma(v)A(v) = \gamma(u)\gamma(v')[A(u)A(v') + B(v')u], \tag{34}$$

$$\frac{B(v)}{A(v)} = \frac{A(v')B(u) + B(v')}{A(u)A(v') + B(v')u}. \tag{35}$$

The third constraint, the isotropy of space-time, results in the requirement that the change of orientations of the variable axis are indistinguishable, provided $u' = -u$. This leads to the parity relations $\gamma(-u) = \gamma(u)$, $A(-u) = A(u)$, and $B(-u) = -B(u)$. The relations are sufficient for the derivation of the Lorenz transformation. The theory of relativity tells us that the velocity of a physical object can not exceed the value of c, the velocity of light in the vacuum. The expression of this statement is the Lorenz transformation which yields the limitation of any velocity. As concerns the relativity principles including the scale degrees of freedom, we need to distinguish two approaches with respect to the concept of motion.

The first one, the scale relativity, is based on the premise that the relation between the energy or momentum of a particle and its velocity is inoperative in fractal space-time. Instead of the motion velocity, the quantities become functions of the 'velocity of scales'. The 'velocity' depends on the fractal characteristics and does not represent any real motion. The special scale relativity concerns the description of physical events with respect to the self-similar scale

structures which are fractals of various fractal dimensions. The corresponding Lorenz-type scale transformations relate the physical quantities expressed in terms of one fractal structure with the quantities given with respect to the other one. Single fractal structures have different anomalous fractal dimensions and play analogous role as the inertial systems in the motion relativity.

The second approach, the scale-motion relativity, preserves relations of the energy and momentum to the velocity. This applies also to small scales where we assume space-time to possess an intrinsic (fractal-like) structure. Our complete change of view of a particle in the corresponding fractal space-time concerns the divergence of the fractal measure representing the length of the particle's trajectory. The degree of revelation of the structures depends on the resolution. For any given resolution ε^{-1}, the non-differentiable fractal space-time F can be approximated by a Riemann space R_ε defined within a differentiable geometry. As pointed in Ref.[17], the family of the Riemann spaces is characterized by metric tensors curvatures of which are expected to fluctuate in a chaotic way. The fluctuations increase with the decreasing scale. For the high resolutions ε^{-1} the approximations to the fractal measure tend to infinity. The corresponding length z_ε of the particle trajectory can be arbitrary large. The propagation of a physical signal along such a trajectory requires the velocities exceeding the value of c, the speed of light in symmetric space-time. Therefore, the application of the principles of relativity to space-time with fractal properties should be treated carefully. The significant characteristics of the fractal spaces is their fluctuating and irregular nature. The corresponding geodesical lines are extremely unpredictable and fragmented at any scale. As a consequence, the isotropy of space-time is clearly broken. This is connected with the breaking of the reflection invariance at the infinitesimal level [22]. The application of the ideas to space-time at small scales leeds us to leave out the constraint (iii) of the reflection invariance, when considering the relativistic transformations (26) and (27). In that case the unknown functions γ, A, and B do not obey the parity relations resulting from the isotropy requirement. Let us combine Eqs. (32), (33), and (34) into the expression

$$A\left(\frac{u+A(u)v'}{1+B(u)v'}\right)=\frac{A(u)A(v')+B(v')u}{1+B(u)v'}. \tag{36}$$

Its solution has the form

$$A(u)=1+2au, \tag{37}$$

provided $B(u)v'=B(v')u$. The condition gives the function $B(u)=u$ with the normalization constant c included already in the definition of the variable u. The solution satisfies Eq. (35) as well. The violation of the space-time reflection invariance is expressed by a non-zero value of a. In terms of the parameter a, the composed velocity (32) can be written as follows

$$v=\frac{v'+u+2auv'}{1+uv'}. \tag{38}$$

Using this relation, Eq. (33) becomes

$$\gamma\left(\frac{v'+u+2auv'}{1+uv'}\right)=\gamma(u)\gamma(v')(1+uv'). \tag{39}$$

Its solution, which for $a=0$ is given by the standard γ factor, has the form

$$\gamma(u)=\frac{1}{\sqrt{1+2au-u^2}}. \tag{40}$$

4.1 Space-time asymmetry in 3+1 dimensions

Let us describe a point P in two Cartesian reference systems S and S'. We assume that the systems are oriented parallel to each other and that S' is moving relative to S with the velocity u in the direction of the positive x-axis. We suppose that the asymmetry expressed by the parameter a is parallel to the velocity u. The relativistic transformations of the coordinates and time are given by

$$x_1' = \gamma(u)\,[x_1 - ut]\,, \qquad x_i' = x_i, \qquad i = 2, 3, \tag{41}$$

$$t' = \gamma(u)\,[(1 + 2au)\,t - ux_1]\,. \tag{42}$$

The inverse relations

$$x_1 = \gamma(u)\,[(1 + 2au)x_1' + ut']\,, \qquad x_i = x_i', \qquad i = 2, 3, \tag{43}$$

$$t = \gamma(u)\,[t' + ux_1'] \tag{44}$$

are obtained as the solution of Eqs. (41) and (42) with respect to the unprimed variables. They can be also derived from the equations by the interchange $\vec{x} \leftrightarrow \vec{x}'$, $t \leftrightarrow t'$, $u \leftrightarrow u'$, and by the relation

$$u' = -\frac{u}{1 + 2au}. \tag{45}$$

This formula connects the velocity u' of the system S in the S' frame with the velocity u of the system S' in the S reference system. Because of the asymmetry parameter a, the magnitudes of the two velocities are not equal. For the vanishing value of a, the transformations turn into the usual relativistic transformations of the coordinates and time. One can show by the direct calculation that the invariant of the transformations has the form

$$t^2 - x_1^2 + 2tax_1. \tag{46}$$

If the space-time anisotropy $\vec{a}$ acquires an arbitrary direction, we write the invariant in the more general form

$$\hat{a}_{\mu\nu}x^\mu x^\nu = t^2 - \vec{x}^2 + 2t\vec{a}\cdot\vec{x} - (\vec{a}\times\vec{x})^2 \equiv \tau^2. \tag{47}$$

Besides the diagonal part, it has extra terms given by a non-zero values of the vector $\vec{a}$. Similar extra terms of the relativistic invariant were considered in Ref. [22] and associated with breaking of the reflection invariance assumed at the infinitesimal level. In Ref. [10], the off-diagonal terms in the metric are connected with distortion of space-time by the recoil particle in the interaction. Using the four dimensional notation, the invariant (47) corresponds to the metric

$$\hat{a}_{\mu\nu} = \left(\begin{array}{cc} -d_{ij} & a_i \\ a_j & 1 \end{array} \right), \qquad d_{ij} = (1 + a^2)\delta_{ij} - a_i a_j. \tag{48}$$

Here the indices i and j numerate the first three rows and columns of the matrix $\hat{a}$, respectively. The δ_{ij} is the Kronecker's symbol. Next we present the explicit form for the relativistic transformations of the coordinates and time in the considered case. They must be linear and homogeneous, preserving the invariant (47). The transformations have to possess an internal group structure required by the principle of relativity. We denote the parameter of the group by the symbol $\vec{u}$. The parameter is the velocity of the system S' with respect to the S

reference frame. In connection with the transformation formulae, it is convenient to introduce the notations

$$\gamma = \frac{1}{\sqrt{(1+\vec{a}\cdot\vec{u})^2 - (1+a^2)u^2}} \tag{49}$$

and

$$g = \frac{(1+\vec{a}\cdot\vec{u})\gamma - 1}{u^2}. \tag{50}$$

Here $a^2 = \vec{a}\cdot\vec{a}$ and $u^2 = \vec{u}\cdot\vec{u}$. Let us define the following combinations of γ and g,

$$\gamma_\pm = gu^2 \pm \gamma\vec{a}\cdot\vec{u}, \qquad g_\pm = \gamma(1+a^2) \pm g\vec{a}\cdot\vec{u}. \tag{51}$$

Now we are ready to consider the relations

$$\vec{x}' = \vec{x} - \vec{u}\left[\gamma(t+\vec{a}\cdot\vec{x}) - g\vec{u}\cdot\vec{x}\right], \tag{52}$$

$$t' = t + \left[\gamma_+(t+\vec{a}\cdot\vec{x}) - g_+\vec{u}\cdot\vec{x}\right]. \tag{53}$$

They generalize the special transformations (41) and (42) which are recovered by $\vec{u} = (u,0,0)$ and $\vec{a} = (a,0,0)$. The inverse relations can be obtained by the interchange $\vec{x} \leftrightarrow \vec{x}'$, $t \leftrightarrow t'$, $\vec{u} \leftrightarrow \vec{u}'$, and by the substitution

$$\vec{u}' = -\frac{\vec{u}}{1+2\vec{a}\cdot\vec{u}}. \tag{54}$$

According to the substitution, there exist the symmetry properties

$$\gamma(\vec{u}') = (1+2\vec{a}\cdot\vec{u})\gamma(\vec{u}), \qquad \gamma_\pm(\vec{u}') = \gamma_\mp(\vec{u}), \tag{55}$$

$$g(\vec{u}') = (1+2\vec{a}\cdot\vec{u})^2 g(\vec{u}), \qquad g_\pm(\vec{u}') = (1+2\vec{a}\cdot\vec{u})g_\mp(\vec{u}). \tag{56}$$

Exploiting the properties, the inverse relations

$$\vec{x} = \vec{x}' + \vec{u}\left[\gamma(t'+\vec{a}\cdot\vec{x}') + g\vec{u}\cdot\vec{x}'\right], \tag{57}$$

$$t = t' + \left[\gamma_-(t'+\vec{a}\cdot\vec{x}') + g_-\vec{u}\cdot\vec{x}'\right] \tag{58}$$

with respect to Eqs. (52) and (53) follow immediately. We express the relativistic transformations in a more compact form

$$x' = D(\vec{u})x, \tag{59}$$

where

$$D(\vec{u}) = \begin{pmatrix} \delta_{ij}+gu_iu_j-\gamma u_ia_j & -\gamma u_i \\ -g_+u_j+\gamma_+a_j & 1+\gamma_+ \end{pmatrix}. \tag{60}$$

The inverse matrix reads

$$D(\vec{u}') = D^{-1}(\vec{u}) = \begin{pmatrix} \delta_{ij}+gu_iu_j+\gamma u_ia_j & +\gamma u_i \\ +g_-u_j+\gamma_-a_j & 1+\gamma_- \end{pmatrix}. \tag{61}$$

The transformation matrices can be decomposed into the product

$$D(\vec{u}) = A_x^{-1}(\vec{a})\Lambda(\vec{\beta})A_x(\vec{a}). \tag{62}$$

Here

$$A_x(\vec{a}) = \begin{pmatrix} \sqrt{1+a^2}\delta_{ij} & 0 \\ a_j & 1 \end{pmatrix} \tag{63}$$

and

$$\Lambda(\vec{\beta}) = \begin{pmatrix} \delta_{ij}+g_0\beta_i\beta_j & -\gamma_0\beta_i \\ -\gamma_0\beta_j & \gamma_0 \end{pmatrix}, \tag{64}$$

with

$$\gamma_0 = \frac{1}{\sqrt{1-\beta^2}}, \qquad g_0 = \frac{\gamma_0 - 1}{\beta^2}. \tag{65}$$

The matrix Λ depends on the vector

$$\vec{\beta} \equiv \vec{\beta}_{\vec{u}} = \sqrt{1+a^2}\frac{\vec{u}}{1+\vec{a}\cdot\vec{u}}. \tag{66}$$

Let us notice that the interchange $\vec{u} \leftrightarrow \vec{u}'$ is equivalent to the symmetry $\vec{\beta} \leftrightarrow -\vec{\beta}$. The relativistic transformations (59) preserve the invariant (47). This follows from the relation

$$D^\dagger(\vec{u})\hat{a}D(\vec{u}) = \hat{a} = A_x^\dagger \eta A_x, \tag{67}$$

where η stands for the diagonal matrix η=diag(-1,-1,-1,+1).

The transformations comply the principle of relativity. Mathematically it is expressed by their group properties. Let $D(\vec{u})$ and $D(\vec{v}')$ be two successive relativistic transformations represented by the matrices (60). The composition of the transformations has the form

$$\Omega_x(\vec{\phi})D(\vec{v}) = D(\vec{v}')D(\vec{u}), \tag{68}$$

provided

$$\vec{v} = \frac{\vec{v}' + \vec{u}\,[\gamma(1+\vec{a}\cdot\vec{v}') + g\vec{u}\cdot\vec{v}']}{1+\gamma_-(1+\vec{a}\cdot\vec{v}') + g_-\vec{u}\cdot\vec{v}'}. \tag{69}$$

One can obtain the above relations by exploiting the decomposition (62) and using the structure of the Lorenz group expressed by the formula

$$R(\vec{\phi})\Lambda(\vec{\beta}_v) = \Lambda(\vec{\beta}_{v'})\Lambda(\vec{\beta}_u). \tag{70}$$

The matrix

$$R(\vec{\phi}) = \begin{pmatrix} R^\varphi_{ij} & 0 \\ 0 & 1 \end{pmatrix}, \qquad \vec{\phi} = \vec{v}' \times \vec{u} \tag{71}$$

describes the Thomas precession [23] around the vector $\vec{\phi}$ known in the theory of relativity. The angle of the precession φ depends on the vectors $\vec{\beta}_{v'}$ and $\vec{\beta}_u$. It remains to identify

$$\Omega_x(\vec{\phi}) = A_x^{-1}R(\vec{\phi})A_x \tag{72}$$

and we get Eq. (68). The relativistic transformation of the coordinates and time with rotation of the coordinate axes has the structure

$$D(\vec{u})\Omega_x(\vec{\phi}) = \begin{pmatrix} R^\varphi_{ij}+gu_iu_rR^\varphi_{rj}-\gamma u_ia_j & -\gamma u_j \\ -g_+u_rR^\varphi_{rj}-a_rR^\varphi_{rj}+(1+\gamma_+)a_j & 1+\gamma_+ \end{pmatrix}, \tag{73}$$

provided the asymmetry of space-time is expressed by the vector $\vec{a}$. As concerns Eq. (69), it can be obtained from the usual relativistic composition of the factors $\vec{\beta}$ given by Eq. (66). The inverse relation

$$\vec{v}' = \frac{\vec{v} - \vec{u}\,[\gamma(1+\vec{a}\cdot\vec{v}) - g\vec{u}\cdot\vec{v}]}{1 + \gamma_+(1+\vec{a}\cdot\vec{v}) - g_+\vec{u}\cdot\vec{v}} \tag{74}$$

corresponds to the composition of the transformations in the following form

$$\Omega_x(-\vec{\phi})D(\vec{v}') = D(\vec{v})D^{-1}(\vec{u}). \tag{75}$$

When using Eqs. (69) and (74), we get

$$1 + \vec{a}\cdot\vec{v} = \gamma\frac{(1+\vec{a}\cdot\vec{u})(1+\vec{a}\cdot\vec{v}') + (1+a^2)\vec{u}\cdot\vec{v}'}{1 + \gamma_-(1+\vec{a}\cdot\vec{v}') + g_-\vec{u}\cdot\vec{v}'}, \tag{76}$$

$$1 + \vec{a}\cdot\vec{v}' = \gamma\frac{(1+\vec{a}\cdot\vec{u})(1+\vec{a}\cdot\vec{v}) - (1+a^2)\vec{u}\cdot\vec{v}}{1 + \gamma_+(1+\vec{a}\cdot\vec{v}) - g_+\vec{u}\cdot\vec{v}}. \tag{77}$$

It follows from the relations that

$$\gamma(\vec{v}) = \gamma(\vec{v}')\,[1 + \gamma_-(1+\vec{a}\cdot\vec{v}') + g_-\vec{u}\cdot\vec{v}']\,, \tag{78}$$

$$\gamma(\vec{v}') = \gamma(\vec{v})\,[1 + \gamma_+(1+\vec{a}\cdot\vec{v}) - g_+\vec{u}\cdot\vec{v}]\,. \tag{79}$$

The formulae generalize Eq. (39). Region of the accessible values of the velocities is given by the factor γ. The boundary of the region is fixed by the condition $\gamma(\vec{v}) = \infty$. For a given value of $\vec{a}$, it is an ellipsoid

$$(v_\parallel - a)^2 + (1+a^2)v_\perp^2 = 1 + a^2 \tag{80}$$

in the velocity space. The focus of the ellipsoid is situated into the point $\vec{v} = 0$. The $v_\parallel$ and $v_\perp$ denote the velocity components which are parallel and perpendicular to the vector $\vec{a}$, respectively. The ellipsoid is invariant under the relativistic transformations (69) and (74). In this case of $\vec{u} = (u,0,0)$ and $\vec{a} = (a,0,0)$, the composition of the velocities has the simple form

$$v_1' = \frac{v_1 - u}{1 + 2au - uv_1}, \qquad v_i' = v_i\frac{\sqrt{1+2au-u^2}}{1+2au-uv_1}, \qquad i = 2,3. \tag{81}$$

The inverse relations can be obtained by the interchange $\vec{v} \leftrightarrow \vec{v}'$ and $u \leftrightarrow u'$. Using Eq. (45), they can be written as follows

$$v_1 = \frac{v_1' + u + 2auv_1'}{1 + uv_1'}, \qquad v_i = v_i'\frac{\sqrt{1+2au-u^2}}{1+uv_1'}, \qquad i = 2,3. \tag{82}$$

4.2 Energy and momentum

Consider a material particle in space-time. In relativistic mechanics, the position and momentum of the particle are given by the four-vectors $x^\mu = \{\vec{x}, t\}$ and $p^\mu = \{\vec{P}, E\}$, respectively. Let us define an 'elementary' particle as an object which reveals no internal structure at any resolution considered. We comprehend the notion of elementarity as a relative concept which

relies on the scales we are dealing with. For the infinite resolution it should be a perfect point whose trajectory is a fractal curve. For an arbitrary small but still finite resolution ε^{-1} the perfect point is approximated by a particle which we call 'elementary' with respect to this resolution. It is therefore natural to assume that the concepts of the momentum, energy, mass and the velocity of the 'elementary' particle have good physical meaning also at the scales where space-time is expected to break down its isotropy.

We denote the values of the momentum and the energy of the elementary particle by ($\vec{P}$ and E) or ($\vec{P}'$ and E') in the reference systems S or S', respectively. In consistence with the principle of relativity and the ideas presented above, we search for relations connecting these quantities. In order to do that, let us first define associative variables $\pi^\mu = \{\vec{\pi}, \pi_0\}$ with the following property. The components of the variables determined relative to the systems S and S' transform in the way

$$\pi' = \Pi(\vec{u})\pi, \tag{83}$$

where

$$\Pi(\vec{u}) = \begin{pmatrix} \delta_{ij}+gu_iu_j-\gamma a_iu_j & -g_+u_i+\gamma_+a_i \\ -\gamma u_j & 1+\gamma_+ \end{pmatrix}. \tag{84}$$

The inverse transformation $\pi = \Pi^{-1}(\vec{u})\pi'$ is given by the matrix

$$\Pi^{-1}(\vec{u}) = \begin{pmatrix} \delta_{ij}+ gu_iu_j+\gamma a_iu_j & +g_-u_i+\gamma_-a_i \\ +\gamma u_j & 1+\gamma_- \end{pmatrix}. \tag{85}$$

There exists mutual correspondence between the transformation matrices Π and D given by the matrix transposition

$$\Pi(\vec{u}) = D^\dagger(\vec{u}). \tag{86}$$

According to the relation, the matrix Π can be expressed in the form

$$\Pi(\vec{u}) = A_\pi^{-1}(\vec{a})\Lambda(\vec{\beta})A_\pi(\vec{a}) \tag{87}$$

where

$$A_\pi^{-1}(\vec{a}) = A_x^\dagger(\vec{a}). \tag{88}$$

The group properties of the transformations (83) are determined with respect to the composition

$$\Omega_\pi(\vec{\phi})\Pi(\vec{v}) = \Pi(\vec{v}')\Pi(\vec{u}), \tag{89}$$

provided the velocities $\vec{u}$, $\vec{v}'$, and $\vec{v}$ satisfy the relation (69). Here

$$\Omega_\pi(\vec{\phi}) = A_\pi^{-1}R(\vec{\phi})A_\pi. \tag{90}$$

We show that Eqs. (68) and (89) are consistent with relation (86). Let us transpose Eq. (68). Exploiting the correspondence (86) and using Eqs. (72), (88), and (90), we can write

$$\Pi(\vec{v})\Omega_\pi(-\vec{\phi}) = \Pi(\vec{v})\Omega_x^\dagger(\vec{\phi}) = \Pi(\vec{u})\Pi(\vec{v}'). \tag{91}$$

We apply the transposition operation on Eq. (70) too. As the matrices Λ are invariant under the operation, we obtain the composition of the parameters $\vec{\beta}_u$ and $\vec{\beta}_{v'}$ in the mutual reverse

order. From the symmetry reasons, the composition must be of the same form as Eq. (70). We have therefore

$$R(-\vec{\phi})\Lambda(\vec{\beta}_w) = \Lambda(\vec{\beta}_v)R(-\vec{\phi}) = \Lambda(\vec{\beta}_u)\Lambda(\vec{\beta}_{v'}). \tag{92}$$

The vector $\vec{\beta}_w$ corresponds to the velocity $\vec{w}$ according to Eq. (66). The velocity is given by the formula (69) in which the velocities $\vec{u}$ and $\vec{v}'$ are mutually interchanged. Multiplying Eq. (92) by the A_π^{-1} from the left and by the A_π from the right, we get

$$\Omega_\pi(-\vec{\phi})\Pi(\vec{w}) = \Pi(\vec{v})\Omega_\pi(-\vec{\phi}). \tag{93}$$

Together with Eq. (91) one has

$$\Omega_\pi(-\vec{\phi})\Pi(\vec{w}) = \Pi(\vec{u})\Pi(\vec{v}'). \tag{94}$$

After performing the interchange $\vec{u} \leftrightarrow \vec{v}'$, we obtain Eq. (89). It was thus shown that the composition of two successive transformations of the variables π follows from the composition of the corresponding transformations of the coordinates and time, provided their transformation matrices are connected by the relation (86).

Unlike the transformations of the coordinates and time, the invariant combination

$$\pi_0^2 - \vec{\pi}^2 + 2\pi_0\vec{a}\cdot\vec{\pi} \tag{95}$$

constructed from the variables π does not correspond to the metrics (48). In order to remove this defect we have to determine the 4-momentum of free particle by means of new variables. The transformations of such variables should preserve the same metric invariant as the transformations of their kinematical counterparts, the coordinates and time. We show that there exists two sets of the variables $p_s^\mu = \{\vec{P}_s, E\}$, $s = L, R$ defined by the relation

$$\pi = A_s(\vec{a})p_s, \tag{96}$$

with

$$A_s(\vec{a}) = \begin{pmatrix} \delta_{ij} \pm \varepsilon_{ijk}a_k & 0 \\ 0 & 1 \end{pmatrix}, \tag{97}$$

which comply the requirement. Here ε_{ijk} is the Levi-Civita symbol. The plus (in the next every upper) sign and the minus (in the next every lower) sign corresponds to $s = L$ and $s = R$, respectively. We will regard the variables p_s^μ as the 4-momentum of an elementary particle in space-time characterized by the asymmetry $\vec{a}$. We attribute the first set of the variables ($s = L$) to the particle which we call left-handed. The second set ($s = R$) corresponds to the particle revealing right-handed type of motion. The relation between the momenta $\vec{P}_s$ and the above considered variable $\vec{\pi}$ reads

$$\vec{\pi} = \vec{P}_s \pm \vec{P}_s\times\vec{a}, \qquad \vec{P}_s = \frac{\vec{\pi} \pm \vec{a}\times\vec{\pi} + (\vec{a}\cdot\vec{\pi})\vec{a}}{1 + a^2}. \tag{98}$$

In the context of these definitions, we introduce the associative variables $\xi_s^\mu = \{\vec{\xi}_s, \xi_0\}$, $s = L, R$, with respect to the coordinates and time by the formula

$$\xi_s = A_s(-\vec{a})x. \tag{99}$$

The transformations of the variables preserve the invariant

$$\xi_0^2 - \vec{\xi}_s^2 + 2\xi_0 \vec{a}\cdot\vec{\xi}_s. \tag{100}$$

Let us introduce the parameters $\vec{U}_s = d\vec{\xi}_s/d\xi_0$. They are related to the velocities $\vec{u}$ as follows

$$\vec{U}_s = \vec{u} \mp \vec{u}\times\vec{a}, \qquad \vec{u} = \frac{\vec{U}_s \mp \vec{a}\times\vec{U}_s + (\vec{a}\cdot\vec{U}_s)\vec{a}}{1+a^2}. \tag{101}$$

Exploiting the additional notations

$$G = \frac{g}{1+a^2}, \qquad G_\pm = \frac{g_\pm}{1+a^2}, \tag{102}$$

the relativistic transformations of the energy/momentum take the form

$$p_s' = \Delta(\vec{U}_s)p_s, \tag{103}$$

where

$$\Delta(\vec{U}) = \begin{pmatrix} \delta_{ij}+GU_iU_j-G_-a_iU_j & GU^2a_i-G_+U_i \\ -\gamma U_j & 1+\gamma_+ \end{pmatrix}, \tag{104}$$

$U^2 = \vec{U}\cdot\vec{U}$. We will skip the index s in the next, where it is insubstantial. Equation (54) implies

$$\vec{U}' = -\frac{\vec{U}}{1+2\vec{a}\cdot\vec{U}}, \tag{105}$$

which together with the symmetry properties (55) and (56) determine the inverse matrix

$$\Delta^{-1}(\vec{U}) = \Delta(\vec{U}') = \begin{pmatrix} \delta_{ij}+GU_iU_j+G_+a_iU_j & GU^2a_i+G_-U_i \\ +\gamma U_j & 1+\gamma_- \end{pmatrix}. \tag{106}$$

The transformation matrixes can be written in the way

$$\Delta(\vec{U}_s) = A_{ps}^{-1}(\vec{a})\Lambda(\vec{\beta})A_{ps}(\vec{a}), \tag{107}$$

where

$$A_{ps}(\vec{a}) = A_\pi(\vec{a})A_s(\vec{a}) = \frac{1}{\sqrt{1+a^2}}\begin{pmatrix} \delta_{ij}\pm\varepsilon_{ijk}a_k & -a_i \\ 0 & \sqrt{1+a^2} \end{pmatrix} \tag{108}$$

and Λ is given by Eq. (64).

The transformations (103) possess group properties. Let us consider two successive transformations expressed by the matrices $\Delta(\vec{U})$ and $\Delta(\vec{V}')$. The resultant transformation is given by

$$\Omega_p(\vec{\phi})\Delta(\vec{V}) = \Delta(\vec{V}')\Delta(\vec{U}), \tag{109}$$

provided

$$\vec{V} = \frac{\vec{V}' + \vec{U}\left[\gamma + G_+\vec{a}\cdot\vec{V}' + G\vec{U}\cdot\vec{V}'\right]}{1+\gamma_- + GU^2\vec{a}\cdot\vec{V}' + G_-\vec{U}\cdot\vec{V}'}. \tag{110}$$

Formula (109) is consequence of Eqs. (70) and (107). The matrix Ω_p has the structure

$$\Omega_p(\vec{\phi}) \equiv \Omega_{ps}(\vec{\phi}) = A_{ps}^{-1}R(\vec{\phi})A_{ps} = A_s^{-1}\Omega_\pi(\vec{\phi})A_s. \tag{111}$$

The inverse relation to Eq. (110) reads

$$\vec{V}' = \frac{\vec{V} - \vec{U}\left[\gamma + G_{-}\vec{a}\cdot\vec{V} - G\vec{U}\cdot\vec{V}\right]}{1 + \gamma_{+} + GU^2\vec{a}\cdot\vec{V} - G_{+}\vec{U}\cdot\vec{V}}. \tag{112}$$

The composition rules (110) and (112) are obtained by substituting Eq. (101) into the formulae (69) and (74), respectively.

The existence of the space-time asymmetry assumed at small scales leads us to the conclusion that the energy-momentum vectors and the space-time positions vectors shall not be treated on the same footing. For a non-zero value of the asymmetry there exist two sets of the mechanical variables p^μ_s, $s = L, R$ and a single set of the kinematical variables x^μ. The variables are defined in space-time characterized by the metrics (48). Single sets of the mechanical variables correspond to the right-handed and left-handed types of motion, respectively. Both of them have either positive or negative energy. The sign of the energy is conserved in whatever reference frame. The 4-vectors x^μ and p^μ_s posses different transformation properties. While the first obey the transformation formula (59), the later are transformed according to Eq. (103). The variables x^μ and p^μ_s are connected by the matrices $A_s(-\vec{a})$ and $A_s(\vec{a})$ with the associated quantities ξ^μ_s and π^μ, respectively. The relativistic transformations of the ξ^μ_s and π^μ preserve the combinations (100) and (95). In the special case, when the velocity $\vec{u}$ is parallel to the vector $\vec{a}$, the difference vanishes, $\vec{u} = \vec{U}_s$ and the transformation matrices have the form

$$D(\vec{u}) = \Delta(\vec{U}) = \begin{pmatrix} \delta_{ij} + \gamma_{-} u_i u_j / u^2 & -\gamma u_i \\ -\gamma u_j & 1 + \gamma_{+} \end{pmatrix}. \tag{113}$$

The inverse matrix reads

$$D^{-1}(\vec{u}) = \Delta^{-1}(\vec{U}) = \begin{pmatrix} \delta_{ij} + \gamma_{+} u_i u_j / u^2 & +\gamma u_i \\ +\gamma u_j & 1 + \gamma_{-} \end{pmatrix}. \tag{114}$$

The transformations (41) - (44) are recovered by putting $\vec{u} = (u, 0, 0)$ and $\vec{a} = (a, 0, 0)$.

Let us now consider the invariant

$$p^2 = \hat{a}_{\mu\nu} p^\mu p^\nu = E^2 - \vec{P}^2 + 2E\vec{a}\cdot\vec{P} - (\vec{a}\times\vec{P})^2 \equiv m_0^2 \tag{115}$$

of the transformations (103). This property follows from the relation

$$\Delta^\dagger(\vec{U}_s)\hat{a}\Delta(\vec{U}_s) = \hat{a} = A^\dagger_{ps}\eta A_{ps}(1 + a^2). \tag{116}$$

The invariant (115) is proportional to the constant m_0, which is the rest mass assigned to a particle in the non-fractal and non-relativistic mechanics. The invariant implies the dependence of the energy of the particle on its momentum in the following way

$$E = \sqrt{(1 + a^2)\vec{P}^2 + m_0^2} - \vec{a}\cdot\vec{P}. \tag{117}$$

We will not consider here the solution with minus sign before the square root corresponding to anti-particles. The energy (117) is positive for arbitrary values of $\vec{a}$ and $\vec{P}$. It has a single minimum for the momentum and energy

$$\vec{P}_0 = M_0\vec{a}, \qquad E(\vec{P}_0) = M_0. \tag{118}$$

The mass M_0 (the minimal energy) depends on the asymmetry parameter $\vec{a}$ by the relation

$$M_0 = \frac{m_0}{\sqrt{1+a^2}}. \tag{119}$$

Beyond the minimum, as the momentum increases, the energy tends to infinity. It consists of two terms. The first term is the free energy

$$\mathcal{E} = \sqrt{\vec{P}^2 + M_0^2} \tag{120}$$

of an object with the mass M_0 scaled by the factor $(1+a^2)^{1/2}$. The second term, $V = -\vec{a}\cdot\vec{P}$, plays the role of a potential induced by the asymmetry of space-time. How can we interpret such a result? The answer was suggested by Nottale. The above energy E includes, a priori, the potential energy contained in the scale structure involved. The structure is already present even in the rest frame of the particle; the rest mass m_0 being itself a geometrical fractal structure of the particle trajectory. The particle may be identified with its own trajectory, which is the fractal-like trajectory of a point-like 'elementary' object moving chaotically with the momentum $\vec{P}$ and having the mass (minimum energy) $M_0(\vec{a}) = E(\vec{P}_0)$. The chaotic nature of the motion is given by the scale dependent fluctuations of the parameter $\vec{a}$. The centre-of-mass frame for the system consisting of the chaotically moving 'elementary' object with the scale dependent mass $M_0(\vec{a})$ is defined by the condition $\vec{P} = 0$. The system represents the counterpart of the 'elementary' object which is the 'dressed' particle with the mass $m_0 = E(\vec{P} = 0)$. Really, the relations (118) and (119) can be inverted and the particle mass

$$m_0 = E_{min}\sqrt{1+a^2} \tag{121}$$

is expressed in terms of the minimal energy $E_{min} = E(\vec{P}_0)$ and the parameter $\vec{a}$. We conjecture that similar considerations concern also other intrinsic characteristics of the particles, such as spin and charge. One can consider the physical quantities as related to the geometrical structures of particle trajectories in the fractal space-time. We anticipate that spin of a particle may be connected to special erratic character of the left-handed or right-handed fractal-like trajectory at small scales. In the domain, where the fractal attributes of the motion expire, the value of $\vec{a}$ diminishes and the fractal dynamics will convert into the relativistic dynamics in smooth space.

We make some comments on the energy momentum conservation. Let us consider a closed system with the mass m_0 which splits into two parts. The decay is governed by the energy momentum conservation, $m_0 = \sqrt{\vec{q}_1^{*2} + m_1^2} + \sqrt{\vec{q}_2^{*2} + m_2^2}$ and $\vec{q}_1^{\,*} = -\vec{q}_2^{\,*}$, as described in the system rest frame. The similar is valid in space-time with broken isotropy. Denoting the energy momentum four-vectors of the decay products by p_1 and p_2, one can write

$$\begin{aligned} m_0^2 &= (p_1+p_2)^2 = m_1^2 + m_2^2 + 2p_1p_2 \\ &= \left[E_1^2 - \vec{P}_1^2 + 2E_1\vec{a}\cdot\vec{P}_1 - (\vec{a}\times\vec{P}_1)^2\right] + \left[E_2^2 - \vec{P}_2^2 + 2E_2\vec{a}\cdot\vec{P}_2 - (\vec{a}\times\vec{P}_2)^2\right] \\ &\quad +2\left[E_1E_2 - \vec{P}_1\cdot\vec{P}_2 + E_1\vec{a}\cdot\vec{P}_2 + E_2\vec{a}\cdot\vec{P}_1 - (\vec{a}\times\vec{P}_1)(\vec{a}\times\vec{P}_2)\right] \\ &= (E_1+E_2)^2 - (\vec{P}_1+\vec{P}_2)^2 + 2(E_1+E_2)\vec{a}\cdot(\vec{P}_1+\vec{P}_2) - \left[\vec{a}\times(\vec{P}_1+\vec{P}_2)\right]^2. \end{aligned} \tag{122}$$

We see that if the four vectors p_1 and p_2 are characterized by the invariant (47), their sum p_1+p_2 possesses this property too. This implies the conservation of the total energy and

momentum, $E = E_1 + E_2$ and $\vec{P} = \vec{P}_1 + \vec{P}_2$, which results in the conservation of the free energy

$$\sqrt{\vec{P}^2 + M_0^2} = \sqrt{\vec{P}_1^2 + M_1^2} + \sqrt{\vec{P}_2^2 + M_2^2} \tag{123}$$

as well. In the center-of-mass system, the decay is characterized by the products with equal anti-parallel momenta ($\vec{P}_1^* = -\vec{P}_2^*$) for arbitrary value of $\vec{a}$. For a given m_0, the momenta are smaller with respect to their values $\vec{q}_i^*$ when $\vec{a} = 0$.

4.3 Relations of the kinematical and mechanical variables

Fundamental concepts of the special theory of relativity lead us to the relation between the energy/momentum of a material particle and its velocity. The velocity is limited within the sphere of the radius c in every system of reference and is oriented in the direction of the particle momentum. This concerns the description of physical events in the homogeneous and isotropic space-time where the relativistic principle requires all systems of inertia to be treated equivalently. We show how the relations change when we abandon the space-time isotropy which we expect to break down at small scales.

First we have to determine how depend the variables π^μ on the velocity $\vec{v}$. In other words, we are searching for the functions $\vec{f}_1$ and f_2,

$$\vec{\pi} = \vec{f}_1(\vec{v}, \vec{a}), \qquad \pi_0 = f_2(\vec{v}, \vec{a}), \tag{124}$$

which are form invariant with respect to the relativistic transformations of the variables π^μ and the velocity $\vec{v}$. One can convince itself that the expressions

$$\vec{\pi} = \left[(1 + a^2)\vec{v} + (1 + \vec{a}\cdot\vec{v})\vec{a}\right] \gamma(\vec{v}) \frac{m_0}{\sqrt{1 + a^2}}, \tag{125}$$

$$\pi_0 = (1{+}\vec{a}\cdot\vec{v})\gamma(\vec{v}) \frac{m_0}{\sqrt{1 + a^2}} \tag{126}$$

fulfill the requirements. Really, substituting the expressions into the transformation formula (83), one arrives at the system

$$\begin{aligned} &\begin{pmatrix} (1{+}a^2)v_i' + (1{+}\vec{a}\cdot\vec{v}')a_i \\ 1 + \vec{a}\cdot\vec{v}' \end{pmatrix} \gamma(\vec{v}') = \\ &= \begin{pmatrix} \delta_{ij} {+} g u_i u_j {-} \gamma a_i u_j & -g_+ u_i {+} \gamma_+ a_i \\ -\gamma u_j & 1{+}\gamma_+ \end{pmatrix} \begin{pmatrix} (1{+}a^2)v_i + (1{+}\vec{a}\cdot\vec{v})a_i \\ 1 + \vec{a}\cdot\vec{v} \end{pmatrix} \gamma(\vec{v}) \end{aligned} \tag{127}$$

consisting of four equations. The last one is an identity. This can be shown by using Eqs. (77) and (79). In the same manner one can convince itself, that the first three equations of the system are consistent with Eq. (74). Now we substitute Eq. (96) into the formulae (125) and (126) and get

$$\vec{P}_s = M\left[\vec{v} + (1{+}2\vec{a}\cdot\vec{v})\vec{a}\right] \mp M(\vec{v}\times\vec{a}), \tag{128}$$

$$E = (1{+}\vec{a}\cdot\vec{v})M. \tag{129}$$

These are the expressions for the momentum and the energy of a left-handed ($s = L$ with upper sign) and right-handed ($s = R$ with lower sign) 'elementary' particle moving with the velocity $\vec{v}$ in space-time characterized by the vector anisotropy $\vec{a}$. As can be seen by direct

calculation, the formulae are consistent with the invariant (47) and (115). The coefficient of the proportionality between the momentum $\vec{P}_s$ and the velocity $\vec{v}$ is denoted by the symbol M and represents the inertial mass of the particle. The inertial mass depends on the velocity in the way

$$M(\vec{v}) = M_0\gamma(\vec{v}). \tag{130}$$

The M_0 is the rest mass of the 'elementary' particle given by Eq. (119). The rest mass corresponds to the minimal energy (118).

Let us now derive the inverse expression with respect to Eq. (128). Besides the invariant (47) and (115), one can construct the invariant relation

$$(A_x^\dagger \eta A_{ps})_{\mu\nu} x^\mu p^\nu = tE - \vec{x}\cdot\vec{P}_s + 2\vec{a}\cdot\vec{x}E \mp \vec{a}\cdot(\vec{x}\times\vec{P}_s) = \tau M_0. \tag{131}$$

It represents the equation of the elementary particle trajectory expressed in terms of its momentum, energy, and its mass M_0. The solution of the equation is $\vec{x} = \vec{v}t$, where

$$\vec{v} = \frac{\vec{P}_s \pm \vec{P}_s\times\vec{a} - \vec{a}E}{(1+2a^2)E - \vec{a}\cdot\vec{P}_s}. \tag{132}$$

Really, if we substitute the solution into Eq. (131) and exploit the formulae (117), (128), and (129), we get

$$t = \tau\gamma. \tag{133}$$

The proportionality relates the time t recorded in a system S to the particle's proper time τ. Equation (132) can be rewritten in a more convenient form

$$M\vec{v} = \frac{\vec{P}_s \pm \vec{P}_s\times\vec{a} - \vec{a}E}{1+a^2}. \tag{134}$$

We see that for the zero value of the momentum there exists the non-zero value of the velocity

$$\vec{v}_0 = -\frac{\vec{a}}{1+2a^2}. \tag{135}$$

The velocity corresponds to the energy $E = m_0$. According to Eq. (115), the constant m_0 is an invariant of the relativistic transformations and represents the rest mass assigned to a particle in the non-fractal (and non-relativistic) limit. Let us stress that, for the energy $E = m_0$, there exist infinite number of the mutually correlated values of $\vec{P}_s$ and $\vec{v}$. They are the momenta and the velocities attributed to the object moving along a fractal-like trajectory and representing the internal structure of the particle itself. In this way, the particle is reduced to and identified with its own trajectory [17]. For a given resolution, we identify the object with the 'elementary' particle possessing the rest mass M_0. The mass depends on the resolution and is defined in terms of the fractal-like curves characterized in a purely geometrical way. According to Eq. (119), its typical value is

$$M_0(<a^2>) = \frac{m_0}{\sqrt{1+<a^2>}}. \tag{136}$$

The symbol $<a^2>$ stands for the average square of the space-time asymmetries revealed at the considered level of the resolution. Other important values of the momentum and velocity giving the energy $E = m_0$ are $\vec{P}_a = 2m_0\vec{a}$ and $\vec{v}_a = \vec{a}$, respectively. The velocity minimizes

the factor γ. As shown in the Appendix B, it corresponds to the minimal length contractions and to the minimal dilatations of time.

In general, for arbitrary energy of a particle, we have shown the following result. It consists of the claim that in space-time with broken isotropy the momentum of the particle is not parallel to its velocity. Approximating the fractal space-time by a family of the spaces R_ε with differentiable geometry, the velocity fluctuates with respect to the momentum in dependence on the stochastic nature of the anisotropy parameter $\vec{a}$. According to the fluctuations, the 'point-like' particle moves around its momentum passing an unpredictable and chaotic trajectory characteristic for fractals. In dependence on the fluctuating anisotropy $\vec{a}$, the velocity of the particle can be arbitrary large. This is connected with a possibility of propagation of physical signals with velocities exceeding the speed of light in the isotropic space-time. The property is, however, compensated by the extreme irregular and random shape of the trajectories along which the signal is mediated. We stress here that the statements are relative and depend on the scale of the observer as well. When 'measuring' the fractal properties of the particle motion, the observer expresses them in terms of its own fractal characteristics being a fractal itself. This is typical for the parameter $\vec{a}$ which is a function of the scale structures of both the observed particle and the observer (see section V.). According to our opinion, the parameter could have relevance to more deeper context of the metric potentials which have relation to the intimate structure of space-time. It may be connected with a 'field of the space-time asymmetry' reflecting the structure at small scales. Existence of the 'field' would result into a disparity between the energy-momentum and the coordinates and time. Here the disparity is demonstrated by the following commutation relation

$$A_{ps}^{\dagger}\eta A_x - A_x^{\dagger}\eta A_{ps} = \left(\begin{array}{cc} \pm\epsilon_{ijk}2a_k & -2a_i \\ 2a_j & 0 \end{array} \right). \tag{137}$$

The commutator is non-zero provided the non-zero value of the field. In the present paper we approximate the field of the space-time asymmetry in terms of the anisotropy vector $\vec{a}$ and consider it as a random and chaotic quantity. As shown in section V., the anisotropy has relevance to the anomalous excess of the topological dimensions. The investigations in this direction require, however, more detailed and fundamental study.

The ideas tackled in this section concern geodesic reference systems in the immediate surroundings of a given point P in the 4-space. The surroundings depend on the resolution we are dealing with. One can introduce such systems in the proximity of every point of the geodesical lines. According to the ideas about fractal properties of space-time at small scales, we characterize the geodesic systems of inertia by the metric tensors $\hat{a}$. The metrics reflects significant property of the fractal structure of space-time which is breaking its isotropy. The structure is revealed in dependence on the level of the resolution. For a given resolution, it is possible to transform away the anisotropy of the space-time locally, exploiting new pseudo-Cartesian coordinates $r^\mu = \{\vec{r}, r_0\}$ and $k^\mu = \{\vec{K}, K_0\}$. We can introduce the variables in the way

$$r = A_x x, \qquad k = A_{ps} p_s. \tag{138}$$

The explicit form of the equations reads

$$\vec{r} = \sqrt{1+a^2}\vec{x}, \qquad r_0 = t + \vec{a}\cdot\vec{x}, \tag{139}$$

$$\vec{K} = \frac{1}{\sqrt{1+a^2}}\left[\vec{P}_s \pm (\vec{P}_s \times \vec{a}) - \vec{a}E\right], \qquad K_0 = E. \tag{140}$$

Unlike the x and p the pseudo-Cartesian variables r and k are functions of the anisotropy $\vec{a}$. Using the variables, one can write the corresponding relativistic invariant in the form

$$r_0^2(\vec{a}) - \vec{r}^2(a) = \tau^2, \qquad K_0^2 - \vec{K}^2(\vec{a}) = M_0^2(a). \tag{141}$$

The space-time anisotropy is thus removable locally but cannot be removed completely, i.e. simultaneously for every point of the 4-space. Hence, we consider the anisotropy at small scales to be the intrinsic property of space-time itself. Its adequate description assumes approaches within a fractal geometry.

5 Interactions of asymmetric fractal systems

The ability of fractals to structure space-time was discussed in Ref. [17]. Such approach gives us possibility to attribute geometrical notions to the structural parameters characterizing fractal trajectories of free particles. We consider one of the parameters to be the scale dependent coefficient $\vec{a}$ reflecting breaking of the space-time isotropy. The quantity is assumed to have stochastic and irregular nature representing the fractal properties of the structures at small distances. The natural question arises whether one can organize a region in which the structures could be somehow oriented. We answer the question positively and argue that such region could be created in the interactions of hadrons and nuclei. This concerns high energies where the objects reveal fractal composition in terms of the parton content involved. The fractality results from non existence of lower cutoff at which the structures would stop. We conjecture that the interactions of the fractal objects affect the character of space-time at small scales. One can imagine that the chaotic character of the space-time anisotropy can be oriented and space-time 'polarized' by the interactions of fractals possessing mutually different anomalous dimensions. In other words, we conjecture that the interactions of the asymmetric fractal systems result in polarization of the (QCD) vacuum. The vacuum fluctuations become oriented forming a region of the space-time asymmetry. We denote the asymmetry corresponding to the region by the vector $\vec{\bar{a}}$. Without the organization, the parameter represent scale dependent random quantity $\vec{a}$. As we will show, the $\vec{\bar{a}}$ can be connected with the anomalous (fractal) dimensions of the interacting fractals.

Let us consider the collision of the asymmetric fractal objects. The need to satisfy the principles of the scale-motion relativity implies replacement of the scale independent physical laws by the scale dependent equations. This concerns the energy and momentum which in the presence of a space-time anisotropy are converted to the variables satisfying the formula (117). We apply the formula to the relations connecting the variables of the recoil particle with the corresponding momentum fractions in the constituent interaction. We infer on fractal character regarding the motion of the particle form the requirements which lead to the relations. They result from the phenomenological analysis of the z scaling variable and concern the minimal resolution ε^{-1} with which one can single out the constituent interaction underlying the production of the inclusive particle m_1. The assumption is reflected by the form of the momentum fractions χ_1 and χ_2 which follows from the condition for the maximum of the coefficient (3). According to the requirement, the recoil particle has the energy E' expressed in the way

$$\frac{2E'}{\sqrt{s}} = \chi_1 + \chi_2 = \sqrt{\omega_1^2 + \mu_1^2} + \sqrt{\omega_2^2 + \mu_2^2} - (\omega_1 - \omega_2)\,. \tag{142}$$

For the sake of simplicity, all masses m_i and M_i are neglected. We identify the energy E' with the expression (117). This gives

$$\sqrt{(1+\bar{a}^2)(\chi_z^2+\chi_\perp^2)} - \bar{a}\chi_z = \sqrt{\omega_1^2+\mu_1^2} + \sqrt{\omega_2^2+\mu_2^2} - (\omega_1-\omega_2)\,. \tag{143}$$

Here we have used the notations

$$\chi_z = \frac{2P_z}{\sqrt{s}} = \frac{P_z}{E'_{max}}, \quad \chi_\perp = \frac{2P_\perp}{\sqrt{s}} = \frac{P_\perp}{E'_{max}}. \tag{144}$$

The symbol $\vec{P}$ represents the momentum of the recoil particle defined relative to the space-time domain of the elementary interaction. Its longitudinal and transversal components with respect to the collision axis are denoted by P_z and $P_\perp$, respectively. In the collision of asymmetric fractal systems, we characterize the domain by the space-time anisotropy $\vec{\bar{a}}=(0,0,\bar{a})$. As follows from the conservation of the free energy (123), Eq. (143) splits into two parts

$$\sqrt{(1+\bar{a}^2)(\chi_z^2+\chi_\perp^2)} = \sqrt{\omega_1^2+\mu_1^2} + \sqrt{\omega_2^2+\mu_2^2}, \tag{145}$$

$$\bar{a}\chi_z = \omega_1-\omega_2. \tag{146}$$

The obtained system for the unknown variables χ_z and $\chi_\perp$ depends on the parameter $\bar{a}$. The variation range of the variables is given by the condition $\chi_1+\chi_2 \le 1$. According to Eqs. (142) and (143), it can be rewritten as follows

$$(\chi_z-\bar{a})^2 + (1{+}\bar{a}^2)\chi_\perp^2 \le 1{+}\bar{a}^2. \tag{147}$$

The χ_z and $\chi_\perp$ are bounded inside the ellipsoid given by the asymmetry $\bar{a}$. If we approach the phase-space limit of the reaction (1), the variables tend to their boundary values

$$\chi_z \to \tilde{\chi}_z = \frac{P_z^{max}}{E'_{max}}, \quad \chi_\perp \to \tilde{\chi}_\perp = \frac{P_\perp^{max}}{E'_{max}}, \tag{148}$$

and satisfy the equation of the ellipsoid. Similar applies for any other particle produced in the elementary interaction. The particle's momentum $\vec{P}$ and energy E' are connected by the dispersion relation (117). In the zero mass approximation, the relation can be expressed in the way

$$\left(\frac{P_z}{E'}-\bar{a}\right)^2 + (1{+}a^2)\left(\frac{P_\perp}{E'}\right)^2 = 1+\bar{a}^2. \tag{149}$$

As follows from Eqs. (142) and (143), it is identical to the equation

$$\left(\frac{\chi_z}{\chi_1+\chi_2}-\bar{a}\right)^2 + (1{+}a^2)\left(\frac{\chi_\perp}{\chi_1+\chi_2}\right)^2 = 1+\bar{a}^2, \tag{150}$$

where

$$\frac{P_z}{E'} = \frac{\sqrt{s}}{2E'}\chi_z = \frac{\chi_z}{\chi_1+\chi_2}, \qquad \frac{P_\perp}{E'} = \frac{\sqrt{s}}{2E'}\chi_\perp = \frac{\chi_\perp}{\chi_1+\chi_2}. \tag{151}$$

The values of $\chi_z/(\chi_1{+}\chi_2)$ are limited within the interval

$$\bar{a}_- \le \frac{\chi_z}{\chi_1+\chi_2} \le \bar{a}_+ \tag{152}$$

with

$$\bar{a}_{\pm} = \bar{a} \pm \sqrt{1+\bar{a}^2}. \tag{153}$$

According to the kinematics of the process, the maximal value of $\chi_z^{max}/(\chi_1+\chi_2) = \bar{a}_+$ should correspond to $\chi_2 = 0$. The minimal value of $\chi_z^{min}/(\chi_1+\chi_2) = \bar{a}_-$ is given by $\chi_1 = 0$. The maximum of $\chi_\perp/(\chi_1+\chi_2) = 1$ should be achieved for $\chi_1 = \chi_2$ and thus for $\chi_z/(\chi_1+\chi_2) = \bar{a}$. The conditions are satisfied by the linear combination

$$\chi_z = (\chi_1+\chi_2)\bar{a} + (\chi_1-\chi_2)\sqrt{1+\bar{a}^2}. \tag{154}$$

Substituting the expression (154) into the relation (146), one arrives at the equation for the asymmetry $\bar{a}$. Its solution which complies the physical requirements on the kinematics of the subprocess reads

$$\bar{a} = \frac{\alpha-1}{2\sqrt{\alpha}}\lambda_c, \tag{155}$$

where

$$\lambda_c = \sqrt{\frac{\lambda_1\lambda_2}{(1-\lambda_1)(1-\lambda_2)}}, \qquad \lambda_c \le 1. \tag{156}$$

Using Eqs. (8), (9), (146), (150) and (155), one can express the variables χ_z and $\chi_\perp$ in a simple form

$$\chi_z = \mu_1 - \mu_2, \qquad \chi_\perp = 2\sqrt{\mu_1\mu_2}. \tag{157}$$

When approaching the phase-space limit, $\lambda_c \longrightarrow 1$ and the value of $\bar{a}$ becomes maximal. In the extreme case, the fractions λ_i approach their boundary values

$$\lambda_1 \longrightarrow \tilde{\lambda}_1 = \cos^2(\theta/2), \qquad \lambda_2 \longrightarrow \tilde{\lambda}_2 = \sin^2(\theta/2). \tag{158}$$

Here θ is the detection angle of the inclusive particle m_1. As seen from Eqs. (7) - (10), this corresponds to the transition

$$\chi_z \longrightarrow \tilde{\chi}_z = \sqrt{\alpha}\sin^2(\theta/2) - \frac{1}{\sqrt{\alpha}}\cos^2(\theta/2), \qquad \chi_\perp \longrightarrow \tilde{\chi}_\perp = \sin\theta. \tag{159}$$

All the expressions are given in terms of the coefficient α which is the ratio of the anomalous fractal dimensions of the colliding objects. The collisions of the asymmetric fractal systems are characterized by the different fractal dimensions and thus with $\alpha \ne 1$. In the considered scenario, it results in creation of the domain in which the isotropy of space-time is violated. The space-time anisotropy in the interaction region is given by the formula (155). If $\alpha = 1$, there is no polarization of space-time induced by the interaction. This corresponds to the collisions of the fractals possessing equal fractal dimensions. Similar situation concerns the interaction of the objects which reveal no fractal-like substructure. The asymmetry $\bar{a}$ becomes non-zero for $\alpha \ne 1$. It changes its sign if $\lambda_1 \leftrightarrow \lambda_2$ and $\alpha \leftrightarrow \alpha^{-1}$, i.g. if the interacting fractals are mutually interchanged. The parameter $\bar{a}$ is the product of the induced asymmetry

$$\bar{a}_0 = \frac{\alpha-1}{2\sqrt{\alpha}} \tag{160}$$

and the factor λ_c. The induced asymmetry of space-time results from the interaction of the fractals characterized by mutually different anomalous (fractal) dimensions. The value of the asymmetry was identified [7] with the space component of the four velocity

$$\frac{\mathcal{V}}{\sqrt{1-\mathcal{V}^2}} = \bar{a}_0. \tag{161}$$

The velocity $\mathcal{V}$ has its origin in the asymmetry of the interaction and vanishes in the collisions of objects which possess equal fractal structures ($\alpha = 1$). It can be expressed by the form

$$\mathcal{V} = \frac{\alpha - 1}{\alpha + 1} \tag{162}$$

representing the velocity of a 'space-time drift' induced by the interaction of the parton fractals. The quantity represents no real motion but characterizes local polarization of the vacuum. The velocity depends on the state of scale of the reference systems and satisfies the scale-relativity composition rule

$$\mathcal{V} = \frac{\mathcal{V}_1 + \mathcal{V}_2}{1 + \mathcal{V}_1\mathcal{V}_2}, \tag{163}$$

provided

$$\alpha = \alpha_1\alpha_2. \tag{164}$$

If we exploit the experimentally established [7] relation $\delta_A = A\delta$, the last equation can be rewritten as follows

$$\frac{A_3}{A_1} = \frac{A_3}{A_2}\frac{A_2}{A_1}. \tag{165}$$

This means that the state of scale of the reference system possesses natural scaling property consisting in the following. If one examines the nucleus A_3 by means of the probe A_2, and then the probe nucleus A_2 by another probe A_1, one arrives at the similar structure as if examining the nucleus A_3 with the probe A_1. Physically, the structures are characterized by the asymmetry in the interactions of the fractals which occurs as a consequence of the richer parton content of one fractal as compared to the other one.

The second factor in Eq. (155), λ_c, represents projection of the induced asymmetry onto the asymmetry value as perceptible from the corresponding resolution ε^{-1}. Diminishing the resolution the observed asymmetry $\bar{a}$ decreases. Let us look at possible manifestations of the asymmetry from the experimental point of view. The necessary condition for the polarization of space-time in the interactions of hadrons and nuclei is the high energy regime where the interacting objects reveal internal fractal-like substructure. As one can conjecture from the analysis of the data on inclusive particle production in view of the z scaling regularity, the regime is expected to set on in the energy region $\sqrt{s_{NN}} \geq 20$ GeV where the z scaling becomes valid. In order to deal with sufficient asymmetry, we have to consider the processes in which the factor λ_c is large enough. This concerns the interactions with large transverse momenta of the observed secondaries. We have estimated the expected asymmetry in the case of the experimentally measured inclusive reactions [24] at 400 GeV proton incoming energy. For the pA interactions $\alpha = A$ and $\lambda_c \simeq E_\perp/(\sqrt{s_{NN}}\sqrt{A})$. The asymmetry was evaluated according to the formula (155) in the most optimistic case of $E_\perp = 7$ GeV. We have obtained the values $\bar{a} \sim 0.09 \div 0.13$ for various target nuclei. The relatively high estimates are rather heuristic and should not be taken literally. One a priori does not know whether the full asymptotic regime with respect to the fractal properties of the interaction is achieved at the considered center-of-mass energy. This may occur at higher energies where, for the given transverse momentum, the projection factor λ_c becomes much smaller. The experimental search for the effect should thus relay on the detection of the particles with still higher momenta. It is connected with difficulties in measurements of small cross sections at which the particles are observable. This concerns also the statistical analysis of an experiment from which one could infer on the existence of the possibility to induce a polarization of space-time.

6 Summary

The questions addressed in the paper concern general properties of the particle production at high energies. The properties are connected with the notions such as locality, self-similarity and fractality in the collisions of hadrons and nuclei. They are manifested mostly in the relativistic regime of local parton interactions which underlie the production of the observed secondaries. In this regime, the description of the inclusive cross sections reveals scaling behavior in dependence on the single variable z.

We have discussed some aspects of the relation between the fractality of the interacting objects and the fractal properties of space-time. It is relevant for small scales where the parton composition of the objects is supposed to reveal a fractal-like substructure. The assumption has fundamental consequence which is breaking of the reflection invariance at the infinitesimal level. Special attention is dedicated to the elaboration of the formalism concerning the relativity in spaces with broken isotropy. Our treatment corresponds to a change in the energy formula in the relativistic case. We have obtained explicit relations between the energy/momentum and the velocity in space-time characterized by the asymmetry $\vec{a}$. In view of these results, increase of stochasticity of the parameter with decreasing scales would result in unpredictable fractal-like motion of particles with respect to their momenta. This implies change of the rest mass M_0 in dependence on the value of $\vec{a}$ as well as possibility of motion with the velocities exceeding the speed of light in isotropic space-time. We have determined the coefficient characterizing the anisotropy of space-time in the interactions of the asymmetric fractal systems. It is expressed in terms of the anomalous dimensions of the fractal objects (hadrons and nuclei) colliding at high energies. The relation is illuminated with respect to the choice of the scaling variable z. The variable z represents a fractal measure proportional to the formation length of a produced particle. The scaling hypothesis states that the differential cross section for the production of the particle depends at high energies on its formation length universally and in an energy independent way. The evolution of the formation process is expressed by the scaling function $H(z)$. The proposed scenario is stressed by the results of our analysis concerning experimental data at high energies. Namely, based on the confrontation of the z scaling scheme with the experimental data, we have shown that the anomalous fractal dimensions for the inclusive production of pions ($\delta \sim 0.8$) and for jets ($\delta \sim 1$) nearly correspond to the relation $D = 1 + \delta = 2$. The relation characterizes fractal dimension of Feynman trajectories and is a direct consequence [17] of the Heisenberg uncertainty relations.

Presented approach to the z scaling shows that the observed regularity can have relevance to fundamental principles of physics at small scales. The general assumptions and ideas discussed here underline need of searching new approaches to physics at ultra-relativistic energies. This concerns better understanding of the micro-physical domain tested by large accelerators of hadrons and nuclei.

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Appendix A

We would like to present some properties which follow from the determination of the variables used in our scheme. The elementary interaction of constituents is characterized by the momentum fractions x_1 and x_2. The relation between the fractions is given by the minimum recoil mass hypothesis in the constituent interaction. The variables are determined in a way to maximize the value of Ω, which gives the minimal resolution ε^{-1}. Each interacting constituent consists from a leading part carrying the momentum fraction λ_i and of a parton 'coat' which is a fractal cloud of tiny partons with the momentum fraction χ_i. What penetrates the cloud is usually determined by the virtuality of a probe and is connected with the resolution. The situation is, however, different as compared to the deep inelastic processes where the 'elementary' interaction is fixed by the kinematical characteristics of the lepton scattering. In the collisions of the composite objects such as hadrons and nuclei, one can, in principle, recognize the interactions of constituents which underlie the production processes, as well. The level of recognition is given by the resolution ε^{-1}, with which one can single out the corresponding subprocesses. It concerns both hard and soft collisions characterized by the different momentum transfer. This in turn determines the virtuality of a probe carrying the momentum transferred and penetrating into the fractal substructure of the very constituents. The squares of the 4-momenta transferred $-Q_1^2$ and $-Q_2^2$ from the first and the second interacting constituent are as follows

$$Q_1^2 = (x_1P_1 - q)^2, \qquad Q_2^2 = (x_2P_2 - q)^2. \tag{166}$$

In the zero mass approximation, the quantities are correlated with the square of the subprocess energy s_x via the relation

$$s_x + Q_1^2 + Q_2^2 = 0. \tag{167}$$

The transferred momenta are usually considered as virtualities of the probes that penetrate the internal structure of the interacting objects. If the underlying interaction of constituents does not posses the contact character, the virtualities are carried by the quanta of the calibration fields. Then the fields mediate the interaction between the constituents. The transferred momenta $-Q_1^2$ and $-Q_2^2$ are connected with resolution. They are equal for

$$\frac{\chi_1}{\chi_2} = \frac{\lambda_1}{\lambda_2}. \tag{168}$$

The condition determines the boundary between the phase space hemispheres [7] belonging to the interacting objects 1 and 2. We have $-Q_1^2 > -Q_2^2$ in the hemisphere corresponding to the object 2. This region is preferable to study of the processes in which the constituents from the object 1 penetrate deeper into the structure of the object 2 and test it in more detail. For $-Q_1^2 < -Q_2^2$ it is vice versa. With the increasing values of $-Q_i^2$, the interaction of the constituents take place on still smaller distances. This in turn increases the spatial resolution necessary for investigation of fractality at small scales.

Next we will show that our determination of the momentum fractions accounts for the back-to-back topology in the constituent's center-of-mass system S_c. First, let us consider the momenta of the inclusive particle and its recoil in the total center-of-mass system S. For the nucleon-nucleon collisions the parameter $\alpha = 1$ and the 'coats' of the interacting constituents carry the momentum fractions

$$\chi_1 \to \bar{\mu}_1 \equiv \lambda\sqrt{\frac{1-\lambda_1}{1-\lambda_2}}, \qquad \chi_2 \to \bar{\mu}_2 \equiv \lambda\sqrt{\frac{1-\lambda_2}{1-\lambda_1}}. \tag{169}$$

The constituents are indistinguishable for $x_1 = x_2$. In this region, each of them possesses the cloud of tiny partons with the same momenta given by $\bar{\mu}_1 = \bar{\mu}_2$. This is not longer valid for $x_1 \neq x_2$. Let us assume that $x_1 > x_2$. It follows from Eqs. (6) and (8) that $\lambda_1 > \lambda_2$ and $\bar{\mu}_1 < \bar{\mu}_2$. This implies the situation when the recoil object moves in the direction not precise opposite to the inclusive particle m_1 in the system S. For the sake of simplicity, we demonstrate this statement in the approximation when all masses are neglected. We use the notations

$$\lambda_1 = \frac{P_2 q}{P_1 P_2} \to \frac{E + q_z}{\sqrt{s}}, \qquad \lambda_2 = \frac{P_1 q}{P_1 P_2} \to \frac{E - q_z}{\sqrt{s}}, \tag{170}$$

$$\bar{\mu}_1 = \frac{P_2 \bar{q}}{P_1 P_2} \to \frac{\bar{E} + \bar{q}_z}{\sqrt{s}}, \qquad \bar{\mu}_2 = \frac{P_1 \bar{q}}{P_1 P_2} \to \frac{\bar{E} - \bar{q}_z}{\sqrt{s}}, \tag{171}$$

$$\chi_1 = \frac{P_2 q'}{P_1 P_2} \to \frac{E' + q'_z}{\sqrt{s}}, \qquad \chi_2 = \frac{P_1 q'}{P_1 P_2} \to \frac{E' - q'_z}{\sqrt{s}}, \tag{172}$$

introducing the energy and momentum for the recoil particle by the symbols $\bar{E}$ and $\bar{q}$ (or E' and q' for $\alpha \neq 1$) in the center-of-mass system S. The angles contained by the momenta $\vec{q}$, $\vec{\bar{q}}$, and $\vec{q}'$ with the collision axis oriented in the direction of motion of the colliding object 1 are given by the expressions

$$\tan(\theta/2) = \sqrt{\frac{\lambda_2}{\lambda_1}}, \qquad \tan(\bar{\theta}/2) = \sqrt{\frac{\bar{\mu}_2}{\bar{\mu}_1}}, \qquad \tan(\theta'/2) = \sqrt{\frac{\chi_2}{\chi_1}}, \tag{173}$$

respectively. The relations $x_1 > x_2$ ($x_1 < x_2$) imply $\theta + \bar{\theta} < \pi$ ($\theta + \bar{\theta} > \pi$). This can be proved as follows. Let e.g. $x_1 > x_2$. It is equivalent to $\lambda_1 > \lambda_2$ and

$$1 > \sqrt{\frac{\lambda_2}{\lambda_1}} \sqrt{\frac{1-\lambda_2}{1-\lambda_1}}. \tag{174}$$

Exploiting Eqs. (169) and (170), we can write $1 > \tan(\theta/2)\tan(\bar{\theta}/2)$, and consequently $\theta + \bar{\theta} < \pi$. The inverse inequalities can be proved equivalently. We have thus shown that in the $2 \to 2$ processes there is perfect back-to-back correlation between the inclusive particle an its recoil in the reference system S only for $x_1 = x_2$. This is valid also for the reactions where the parameter $\alpha \neq 1$. Change of the parameter corresponds to a change of the scale of the reference system and results in changing of the resolution. In the center-of-mass system S, the constituent subprocess reveals back-to-back topology in the special case

$$\cos\theta = \frac{1-\alpha}{1+\alpha}. \tag{175}$$

The situation corresponds to the equal momentum fractions $x_1 = x_2$. On the other hand one gets $\theta + \theta' < \pi$ or $\theta + \theta' > \pi$ for $x_1 > x_2$ or $x_1 < x_2$, respectively. We will show how the relations change in the constituent's center-of-mass system S_c. For $x_1 = x_2$, it coincides with the total center-of-mass frame S. For $x_1 \neq x_2$, the system S_c is determined by the condition $x_1 P_1^c = x_2 P_2^c$ where P_i^c are the momenta of the colliding nuclei in S_c. The momenta P_i^c can be related to the momenta P_i as follows

$$P_1^c = P_1 \sqrt{\frac{x_1}{x_2}}, \qquad P_2^c = P_2 \sqrt{\frac{x_2}{x_1}}. \tag{176}$$

This allows us to write

$$\lambda_1 = \frac{P_2^c q^c}{P_1^c P_2^c} = \frac{E^c + q_z^c}{\sqrt{s}} \sqrt{\frac{x_1}{x_2}}, \qquad \lambda_2 = \frac{P_1^c q^c}{P_1^c P_2^c} = \frac{E^c - q_z^c}{\sqrt{s}} \sqrt{\frac{x_2}{x_1}}. \tag{177}$$

The angles θ_c and θ_c' contained by the momenta $\vec{q}_c$ and $\vec{q}_c'$ with the collision axis can be expressed in the system S_c in the way

$$\tan(\theta_c/2) = \sqrt{\frac{\lambda_2 x_1}{\lambda_1 x_2}}, \qquad \tan(\theta_c'/2) = \sqrt{\frac{\chi_2 x_1}{\chi_1 x_2}}. \tag{178}$$

This implies

$$\tan(\theta_c/2)\tan(\theta_c'/2) = 1 \tag{179}$$

and consequently $\theta_c + \theta_c' = \pi$. Really, the substitution of expressions (178) into Eq. (179) gives

$$\sqrt{\lambda_2 \chi_2}(\lambda_1 + \chi_1) = \sqrt{\lambda_1 \chi_1}(\lambda_2 + \chi_2). \tag{180}$$

It remains to exploit the relation $\lambda_1 \lambda_2 = \chi_1 \chi_2$ and one gets the identity. We have thus shown that our determination of the momentum fractions is consistent with back-to-back topology of the collisions in the center-of-mass systems of the interacting constituents.

Appendix B

In this Appendix we discuss some aspects concerning the relativistic transformations of the energy and the coordinates in space-time with broken isotropy. Reasonable definition of the variables assumes the fulfillment of certain requirements resulting from the proper composition of the velocities. It regards the principle of causality and the constraint on the positivity of the energy. At the end we add some comments on the character of the lengths contractions and the dilatations of time.

According to the special theory of relativity, the values of the particle's velocities are bounded within the sphere $u \leq 1$ in any inertial frame. This is given by the factor γ which for $\vec{a} = 0$ and for the superluminous velocities becomes imaginary. The situation changes if we admit the breaking of the space-time isotropy expressed by the non-zero value of $\vec{a}$. The velocity sphere deforms to an ellipsoid with the focus in the beginning of the velocity space. Center of the ellipsoid is shifted into the point $\vec{u} = \vec{a}$ and its larger axis becomes $\sqrt{1+a^2}$. We illustrate the basic properties of the velocity composition in one dimensional case ($a \equiv a_1$, $u \equiv u_1$). The region of the accessible values of the velocities is determined by the condition

$$a_- \leq u \leq a_+ \tag{181}$$

where

$$a_- = a - \sqrt{1+a^2}, \qquad a_+ = a + \sqrt{1+a^2}. \tag{182}$$

The boundaries a_- and a_+ satisfy the relation $\gamma(a_\pm) = \infty$. We have $-1 \leq a_- \leq 0$ and $1 \leq a_+ \leq \infty$ for $a \geq 0$. The negative values of a imply $-\infty \leq a_- < -1$ and $0 \leq a_+ < 1$. Consider two velocities, u and v'. If the velocities are from the interval $a_- \leq u, v' \leq a_+$, the

composed velocity (38) is bounded by the condition $a_- \leq v \leq a_+$ as well. If the limiting velocity a_- or a_+ is composed with a velocity u,

$$a_- = \frac{a_- + u + 2aa_-u}{1 + a_-u}, \qquad a_+ = \frac{a_+ + u + 2aa_+u}{1 + a_+u}, \tag{183}$$

one gets again a_- or a_+, respectively. As follows from the relations

$$a_+ = -\frac{a_-}{1 + 2aa_-}, \qquad a_- = -\frac{a_+}{1 + 2aa_+}, \tag{184}$$

the limiting velocities a_- and a_+ are mutually inverse with respect to Eq. (45). For $a > 0$, the instant velocity of the particle is bounded from above by the value of a_+ which is larger than unity. This gives possibility of the motion with the velocities exceeding the speed of light c in isotropic space-time.

We will show that propagation of an energetic signal with such velocities fulfills the principle of causality. According to the requirement, the consequence - the detection of a signal can not precede its emission in whatever system of reference. Let us assume that the signal was emitted in the point (x_1, t_1) and detected at (x_2, t_2) with respect to the frame S, $dt = t_2 - t_1 > 0$. The velocity of the signal propagation is $v = dx/dt$, $dx = x_2 - x_1$. Let us look at the two events from the system S' moving relatively to the initial one with the speed u. According to the transformation (42) we have

$$dt' = \gamma(u)\left[(1 + 2au)dt - udx\right] = \gamma(u)dt\,(1 + 2au - uv)\,. \tag{185}$$

The factor on the right hand side is non-negative for any velocities from the interval $a_- \leq u, v \leq a_+$. Consequently $dt' = t_2' - t_1' \geq 0$. The same is valid in the general case when the signal propagates between the points $(\vec{x}_1, t_1)$ and $(\vec{x}_2, t_2)$ with the velocity $\vec{v} = d\vec{x}/dt$. We assume that the system S' is moving with respect to the reference frame S with the velocity $\vec{u}$. As follows from Eqs. (59) and (60), the time interval dt' of the signal propagation relative to the system S' is given by

$$dt' = (1 + \gamma_+)dt + \gamma_+\vec{a}\cdot d\vec{x} - g_+\vec{u}\cdot d\vec{x} = [1 + \gamma_+(1 + \vec{a}\cdot\vec{v}) - g_+\vec{u}\cdot\vec{v}]dt. \tag{186}$$

We see from Eq. (79) that the expression in the brackets is non-negative. This implies $dt' \geq 0$ in agreement with the causality principle, which is not violated in space-time with broken isotropy.

The next step is to prove the positivity of the transmitted energy. Combining Eqs. (125) and (126), one has

$$\vec{\pi} = \frac{(1 + a^2)}{1 + \vec{a}\cdot\vec{v}}\pi_0\vec{v} + \pi_0\vec{a}. \tag{187}$$

We insert this expression into Eq. (83) and after some manipulation write

$$E' = E\frac{\gamma}{1 + \vec{a}\cdot\vec{v}}\left[(1 + \vec{a}\cdot\vec{u})(1 + \vec{a}\cdot\vec{v}) - (1 + a^2)\vec{u}\cdot\vec{v}\right]. \tag{188}$$

When exploiting Eqs. (77) and (79), the relation can be rewritten into the form

$$E' = E\frac{(1 + \vec{a}\cdot\vec{v}')\gamma(\vec{v}')}{(1 + \vec{a}\cdot\vec{v})\gamma(\vec{v})}. \tag{189}$$

As follows from the inequality

$$0 \leq 1 + aa_{-} \leq 1 + \vec{a}\cdot\vec{v} \tag{190}$$

valid for any velocity $\vec{v}$ bounded by the ellipsoid (80), the factors on the right side of the relation (189) are non-negative. The energy of the signal is thus positive in each system of reference S' which is moving relatively to the system S with the velocity $\vec{u}$. The above mentioned properties enable the propagation of physical signals including transportation of the energy with the velocities exceeding the value of c - the speed of light in isotropic space-time. This can occur at small scales within the regions with broken space-time isotropy.

Based on the transformations (59), we can draw conclusions regarding the course of time and the change of lengths of the elementary sections expressed relative to the systems S and S'. Let us consider a clock at rest with respect to the system S'. Time recorded by the clock is referred as the proper time. According to Eq. (133), the increase of the proper time $d\tau$ and the corresponding increase of time dt in the system S are related as follows

$$dt(\vec{v}) = \frac{d\tau}{\sqrt{(1 + \vec{a}\cdot\vec{v})^2 - (1+a^2)v^2}}. \tag{191}$$

In view of the asymmetry represented by the factor $\vec{a}$, the course of the clock time can be even faster than in its rest frame, when observed by a moving observer. The minimal time interval between two events

$$dt(\vec{v}_a) = \frac{d\tau}{\sqrt{1+a^2}} \leq d\tau, \qquad \vec{v}_a = \vec{a}, \tag{192}$$

is recorded from the system S in which the clocks are moving with the velocity $\vec{v}_a$. The clocks are slowing down (their time intervals increase) if their velocity approaches the limit given by Eq. (80).

The change of the length of a section with the velocity is little more complicated, thought its transverse dimensions with respect to the motion are not subjected to any change. Indeed, if the velocity is oriented e.g. in the direction of the x-axis, the y and z components of the coordinates are invariant with respect to the transformation (59). As concerns the longitudinal contractions, we discuss here the simplified situation in which the section has no transverse dimension relative to its motion. It is natural to define the length dl of the section with respect to S as the difference between the simultaneous coordinate values of its end-points. If the section is at rest in the S' system, its rest length is given by $dl_0 = x'_2 - x'_1$. According to the specific situation considered, the both values are connected with the expression

$$dl(\vec{v}) = dl_0\sqrt{(1 + \vec{a}\cdot\vec{v})^2 - (1+a^2)v^2}. \tag{193}$$

The length $dl(\vec{v})$ observed from the system S is maximal if the section moves with the velocity $\vec{v}_a = \vec{a}$. Its value depends on the parameter $\vec{a}$ in the way $dl(\vec{v}_a) = dl_0\sqrt{1+a^2} \geq dl_0$. The contractions of the elementary section with respect to its maximal value $dl(\vec{v}_a)$ increase if the velocity of the section approaches the boundary given by Eq. (80).

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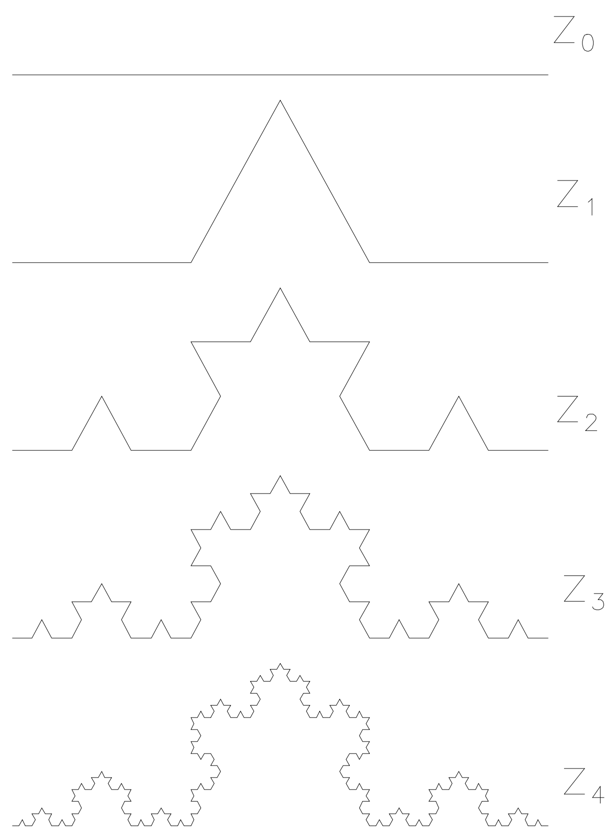

Figure 1: Succesive approximations of the von Koch fractal curve. Its topological dimension is 1, while its fractal dimension is $\ln 4/\ln 3$.